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# Stability of the Triangular Points $L_{4,5}$ in the ER3BP under the Influence of the Octupolar Mass Moment $J_4$ of the Secondary

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***Authors' contributions***

*This work was carried out in collaboration between all authors. Author JS designed the study, wrote the protocol and supervised the work. Author AU performed the numerical computations, analyses and wrote the first draft of the manuscript. Author RS carried out the mathematical computations and managed the literature searches. All authors read and approved the final manuscript.*

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## Abstract

This paper investigates the effects of the quadruple $J_2$ and octupolar mass moment $J_4$ of the secondary on the triangular points $L_{4,5}$ in the elliptic restricted three-body problem (ER3BP). The positions and stability of the triangular points are affected by the perturbations in the shape of the smaller primary (oblateness up to $J_4$) and the elliptic nature of the orbits. An application of the results to double white dwarf binaries, reveals that the triangular points of the binaries are unstable due to the mass ratio μ falling outside the stability range $0 < \mu < \mu_c$; where $(\mu \le 1/2)$.

*Keywords: Celestial mechanics; triangular points; ER3BP.*

## 1 Introduction

One of the most important achievements of mankind is space activity; it makes possible communications, exploration of Earth resources, weather forecast, accurate positioning and other tasks that are part of our

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lives today. Space dynamics plays a very important role in these developments and has been made possible by the restricted three-body problem (R3BP). The R3BP describes the motion of an infinitesimal mass moving under the gravitational effects of two finite-spherical masses, called the primaries, which move in circular orbits around their common barycenter [1] on account of their mutual gravitational attraction with the infinitesimal mass being influenced but not influencing the motion of the primaries. It has received attention of researchers especially in the two and three dimensional cases and with respect to its five equilibrium points, i.e. the collinear (or "Eulerian") points $L_1, L_{2,} L_3$ and the two isosceles triangular (or "Lagrangian") points $L_{4,} L_5$ [2-12]. Also, in general relativity, the3B problem has been the subject of several researches [13-15].

When the orbits of the primaries are elliptic, called the ER3BP, the smaller primary orbits the larger one in an elliptical orbit described by the two-body problem. Here, a non-uniformly rotating-pulsating coordinate system is commonly used with the property that, the positions of the primaries are fixed; however, the Hamiltonian is explicitly time-dependent [16]. In one such study with constant coefficient, [17], the motion of a dust grain particle under the influence of a dark degenerate primary and luminous secondary in the ER3BP. In their study using the binary systems Cen X-4 and RXJ 0450.1-5856 was investigated both numerically and analytically the existence and stability of such systems without taking into consideration the P–R effect was studied.

In the last decades, many different authors consider various perturbing agents such as oblateness, triaxiality, radiation pressure(s) of the primaries, Coriolis and centrifugal forces [5,6,18,19], variation of the masses of the primaries and the infinitesimal mass [20] in the study of the R3BP. Taking into consideration that one or both of the primaries are oblate spheroid, which affects the existence and stability of the equilibrium points, as in the cases of [21,22]; where both primaries are oblate spheroid, [23]; where only the bigger primary is an oblate spheroid; the range of stability for the triangular points increases or decreases depending on the sign of a parameter which depends on the perturbed functions.. Extension of studies went further on the increase in degree of oblateness, [10]; where the effects of oblateness up to $J_4$; of both primaries; together with gravitational potential from a circular cluster of material points on the stability of the triangular equilibrium points in the CR3BP is studied; [24] extended the work of [25] by considering the shape of the second primary as an oblate spheroid with oblateness coefficients up to the second zonal harmonics. [26] considered the influence of even zonal harmonic parameters up to $J_4$ for both primaries, on the existence of the libration points and their linear stability as well as analyzing the existence of periodic orbits around these points.

In this paper, using the double white dwarfs NLTT 11748, LP400-22 and J1257+5428, under the assumption of the sphericity of the primary and oblateness of the secondary, we show the effect of the octupolar mass moment ($J_4$) of the companion star on the stability of the triangular equilibrium points in the framework of the ER3BP.

This work is organized as follow: Section 1 gives the introduction; while Sections 2 and 3 presents the equation of motion for the problem under consideration and locates the triangular equilibrium points; while Section 4 examines the stability of these equilibrium points; finally, the conclusions are drawn in Section 5.

# 2 Equations of Motion

We present the equations of motion of the ER3BP following [10,22] in a synodic- pulsating dimensionless coordinate system, with axes that expand and shrink, considering the secondary an oblate spheroid, with oblateness up to $J_4$, as

$$\xi'' - 2\eta' = \Omega_\xi \; ; \quad \eta'' + 2\xi' = \Omega_\eta ; \qquad \zeta'' = \Omega_\zeta \tag{1}$$

where the force function,

$$\Omega = \frac{1}{(1-e^2)^{½}}\left[\frac{1}{2}\left(\xi^2+\eta^2\right)+\frac{1}{n^2}\left\{\frac{(1-\mu)}{r_1}+\frac{\mu}{r_2}+\frac{\mu B_1}{2r_2^3}-\frac{3\mu B_2}{8r_2^5}\right\}\right] \tag{2}$$

and the mean motion

$$n^2 = \frac{(1+e^2)^{\frac{1}{2}}}{a\,(1-e^2)}\left(1+\frac{3}{2}B_1-\frac{15}{8}B_2\right) \tag{3}$$

The distance of the third body from the primary and secondary are:

$$r_i^2 = (\xi-\xi_i)^2+\eta^2+\zeta^2 \text{ (i=1,2) with } \xi_1 = -\mu, \xi_2 = 1-\mu \tag{4}$$

and $0 < \mu = \frac{m_2}{m_1+m_2} < ½$ is the mass ratio with $m_1, m_2$ as the masses of the primaries positioned at the points $(\xi_{i,}0,0) i = 1,2; B_1$ & $B_2$ are their oblateness and octupolar mass moment $(J_4)$ coefficients, $B_i = J_{2i}R_2^2$ (i=1,2) characterize the oblateness of the smaller primary of mean radius $R_2$ and quadruple and octupolar mass moments (Zonal Harmonic Co-efficient) $J_2$ and $J_4$ respectively; while $a$ and $e$ are respectively the semi-major axis and eccentricity of the orbits.

# 3 Positions of Triangular Points

The equilibrium solutions of the problem are obtained by equating all velocities and acceleration components of the dynamical systems to zero. That is, the equilibrium points are the solutions of the equations:

$$\Omega_\xi = \Omega_\eta = \Omega_\zeta = 0$$

i.e.

$$\begin{aligned}
&\xi - \frac{1}{n^2}\left\{\frac{(1-\mu)(\xi-\xi_1)}{r_1^3}+\frac{\mu(\xi-\xi_2)}{r_2^3}+\frac{3\mu B_1(\xi-\xi_2)}{2r_2^5}-\frac{15\mu(\xi-\xi_2)B_2}{8r_2^7}\right\} = 0\\
&\eta - \frac{1}{n^2}\left\{\frac{(1-\mu)\eta}{r_1^3}+\frac{\mu\eta}{r_2^3}+\frac{3\mu B_1\eta}{2r_2^5}-\frac{15\mu B_2\eta}{8r_2^7}\right\} = 0\\
&\left[\frac{\zeta}{n^2}\left\{\frac{(1-\mu)}{r_1^3}+\frac{\mu}{r_2^3}+\frac{3\mu B_1}{2r_2^5}-\frac{15\mu B_2}{8r_2^7}\right\}\right] = 0
\end{aligned} \tag{5}$$

The positions of the triangular points are obtained from the first two equations of equation (5) above with $\eta \neq 0$ and $\zeta = 0$. From which;

$$n^2 = \frac{1}{r_2^3}+\frac{3B_1}{2r_2^5}-\frac{15B_2}{8r_2^7} \tag{6}$$

and $r_i^3 = \frac{1}{n^2}$, when oblateness of the smaller primary is absent i.e. $B_1 = B_2 = 0$, we have $r_i^3 = \frac{1}{n^2}$, (i=1,2), when the oblateness is considered the value of $r_2$ will change slightly by $\varepsilon$, say

$$r_2 = \varepsilon + \left(\frac{1}{n}\right)^{\frac{2}{3}} \tag{7}$$

Neglecting second and higher order terms of $e^2, B_1, B_2$ and their product, equation (3) becomes

$$n^2 = \frac{1}{a}[1 + \frac{3}{2}e^2 + \frac{3}{2}B_1 - \frac{15}{8}B_2] \tag{8}$$

and then the second equations of (6) gives

$$r_1 = (a)^{1/3}(1 - \frac{e^2}{2} - \frac{B_1}{2} + \frac{5}{8}B_2); \tag{9}$$

From (6), (7) & (8) and neglecting higher order terms, we obtain;

$$\varepsilon = \frac{B_1}{2}(a^{-1/3} - a^{1/3}) - \frac{5}{8}B_2(a^{-1} - a^{1/3}) \tag{10}$$

Substituting $\varepsilon$ into (7) as appropriate, we obtain;

$$r_2{}^2 = a^{\frac{2}{3}}[1 - e^2 + B_1(a^{-\frac{2}{3}} - 1) - \frac{5}{4}B_2\left(a^{\frac{-4}{3}} - 1\right)$$

and (9), becomes;

$$r_1{}^2 = (a)^{\frac{2}{3}}(1 - e^2 - B_1 + \frac{5}{4}B_2) \tag{11}$$

Using 4 & 11, we get

$$\xi = \frac{1}{2} - \mu - \frac{B_1}{2} + \frac{5}{8}a^{-\frac{2}{3}}B_2$$

and

$$\eta^2 = a^{\frac{2}{3}} - \frac{1}{4} - a^{\frac{2}{3}}e^2 + B_1\left(\frac{1}{2} - a^{\frac{2}{3}}\right) + \frac{5}{4}B_2(a^{\frac{2}{3}} - \frac{1}{2}a^{-\frac{2}{3}}) \tag{12}$$

The co-ordinates $(\xi, \pm\eta)$ obtained in equation (12) are the triangular libration points and are denoted by$L_{4,5}$. Using equation (12), for various values of oblateness$B_1$& $B_2$, we compute numerically the positions of the triangular points using the numerical data in Table 1 [18] to show the effects of $B_1$& $B_2$ (Tables 2, 4 and 6), eccentricity e and semi-major axis a (Tables 3, 5 and 7). These effects are shown graphically in Figs. 1-3.

**Table 1. Numerical data**

| **Binary system** | $M_1$ | $M_2$ | $\mu$ |
|---|---|---|---|
| NLTT 11748 | 0.15 | 0.71 | 0.1744 |
| LP400-22 | 0.19 | >0.48 | 0.2836 |
| J1257+5428 | 0.20 | >0.95(0.98) | 0.16949 |

**Table 2. Effects of $B_1$ and $B_2$ for e=0.6, a=0.8; $B_2$=-0.00001 and $B_1$=0.1on LP400-22**

| $B_1$ | $\xi$ | $\pm\eta$ | $B_2$ | $\xi$ | $\pm\eta$ |
|---|---|---|---|---|---|
| 0.001 | 0.215893 | 0.548790 | -0.000001 | 0.166399 | 0.515129 |
| 0.01 | 0.211393 | 0.545815 | -0.00001 | 0.166393 | 0.515126 |
| 0.1 | 0.166393 | 0.515126 | -0.0001 | 0.166327 | 0.515095 |
| 0.2 | 0.116393 | 0.478724 | -0.001 | 0.165675 | 0.514787 |
| 0.3 | 0.0663927 | 0.439317 | -0.01 | 0.159148 | 0.511701 |

**Table 3. Effects of eccentricity and semi-major axis for e=0.6; a=0.8; $B_1$=0.01 and $B_2$=-0.00001 on LP400-22**

| *e* | *ξ* | ±*η* | *A* | *ξ* | ±*η* |
|---|---|---|---|---|---|
| 0.99 | 0.211393 | 0.486284i | 0.99 | 0.211394 | 0.617079 |
| 0.9 | 0.211393 | 0.299807i | 0.9 | 0.211393 | 0.585032 |
| 0.8 | 0.211393 | 0.237944 | 0.8 | 0.211393 | 0.545815 |
| 0.7 | 0.211393 | 0.431142 | 0.7 | 0.211392 | 0.501671 |
| 0.6 | 0.211393 | 0.545815 | 0.6 | 0.211391 | 0.450742 |
| 0.5 | 0.211393 | 0.626665 | 0.5 | 0.211390 | 0.389714 |
| 0.4 | 0.211393 | 0.685761 | 0.4 | 0.211388 | 0.311482 |
| 0.3 | 0.211393 | 0.728418 | 0.3 | 0.211386 | 0.193227 |
| 0.2 | 0.211393 | 0.757418 | 0.2 | 0.211382 | 0.17184i |
| 0.1 | 0.211393 | 0.774296 | 0.1 | 0.211371 | 0.330521i |

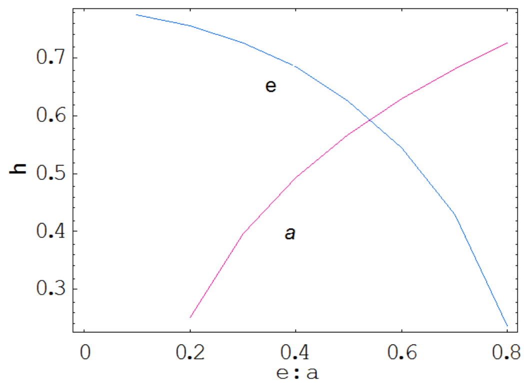


**Fig. 1. Effects of eccentricity and Semi-major axis on the triangular points ofNLTT11748**

**Table 4. Effects of $B_1$ and$B_2$ for e=0.8, a=0.7; $B_1$=0.1 and $B_2$=-0.00001 on J1257+5428**

| $B_1$ | ξ | ±η | $B_2$ | ξ | ±η |
|---|---|---|---|---|---|
| 0.001 | 0.330002 | 0.183096 | -0.000001 | 0.280509 | 0.0705473 |
| 0.01 | 0.325502 | 0.175866 | -0.00001 | 0.280502 | 0.0705350 |
| 0.1 | 0.280502 | 0.070535 | -0.0001 | 0.280431 | 0.0704120 |
| 0.2 | 0.230502 | 0.154474i | -0.001 | 0.279717 | 0.0691695 |
| 0.3 | 0.180502 | 0.229564i | -0.01 | 0.272582 | 0.0552282 |

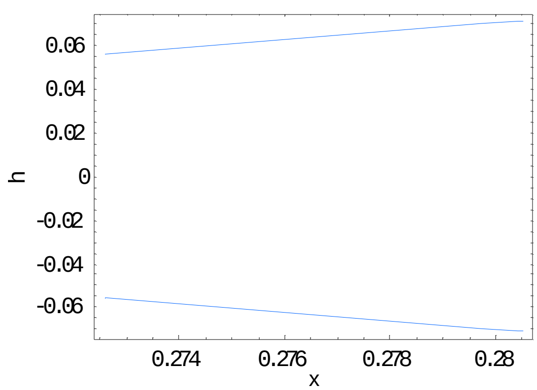


**Fig. 2. Effect of $B_2$ on $L_{4,5}$ of J1257+1548 for $B_1$=0.1, e=0.8 and a=0.7**

**Table 5. Effects of eccentricity and semi-major axis for e=0.8; a=0.7; $B_1$=0.01 and $B_2$=-0.00001 on J1257+5428**

| $e$ | $\xi$ | $\pm\eta$ | $a$ | $\xi$ | $\pm\eta$ |
|---|---|---|---|---|---|
| 0.99 | 0.325502 | 0.487029i | 0.99 | 0.325504 | 0.320401 |
| 0.9 | 0.325502 | 0.321084i | 0.9 | 0.325503 | 0.285052 |
| 0.8 | 0.325502 | 0.175866 | 0.8 | 0.325503 | 0.237944 |
| 0.7 | 0.325502 | 0.386245 | 0.7 | 0.325502 | 0.175866 |
| 0.6 | 0.325502 | 0.501671 | 0.6 | 0.325501 | 0.0631065 |
| 0.5 | 0.325502 | 0.581717 | 0.5 | 0.325500 | 0.156562i |
| 0.4 | 0.325502 | 0.639803 | 0.4 | 0.325498 | 0.234491i |
| 0.3 | 0.325502 | 0.681567 | 0.3 | 0.325496 | 0.296888i |
| 0.2 | 0.325502 | 0.709896 | 0.2 | 0.325492 | 0.35396i |
| 0.1 | 0.325502 | 0.726364 | 0.1 | 0.325481 | 0.411787i |

**Table 6. Effects of $B_1$ and$B_2$ for e=0.3, a=0.7;$B_1$=0.1 and $B_2$=-0.00001 on NLTT 11748.**

| $B_1$ | $\xi$ | $\pm\eta$ | $B_2$ | $\xi$ | $\pm\eta$ |
|---|---|---|---|---|---|
| 0.001 | 0.3256 | $\sqrt{3}/2$ | -0.000001 | 0.320599 | 0.681569 |
| 0.01 | 0.325092 | 0.683469 | -0.00001 | 0.320592 | 0.681567 |
| 0.1 | 0.320592 | 0.681567 | -0.0001 | 0.325021 | 0.681555 |
| 0.2 | 0.275592 | 0.662254 | -0.001 | 0.319807 | 0.681428 |
| 0.3 | 0.225592 | 0.640112 | -0.01 | 0.312672 | 0.680154 |

**Table 7. Effects of eccentricity and semi-major axis for e=0.3; a=0.8; $B_1$=0.01 and $B_2$=-0.00001 on NLTT11748**

| $e$ | $\xi$ | $\pm\eta$ | $a$ | $\xi$ | $\pm\eta$ |
|---|---|---|---|---|---|
| 0.99 | 0.320593 | 0.486284i | 0.99 | 0.320594 | 0.805595 |
| 0.9 | 0.320593 | 0.299807i | 0.9 | 0.320593 | 0.77068 |
| 0.8 | 0.320593 | 0.237944 | 0.8 | 0.320593 | 0.728418 |
| 0.7 | 0.320593 | 0.431142 | 0.7 | 0.320592 | 0.681567 |
| 0.6 | 0.320593 | 0.545815 | 0.6 | 0.320591 | 0.628682 |
| 0.5 | 0.320593 | 0.626665 | 0.5 | 0.320590 | 0.567421 |
| 0.4 | 0.320593 | 0.685761 | 0.4 | 0.320588 | 0.493558 |
| 0.3 | 0.320593 | 0.728418 | 0.3 | 0.320586 | 0.397913 |
| 0.2 | 0.320593 | 0.757418 | 0.2 | 0.320582 | 0.250619 |
| 0.1 | 0.320593 | 0.774296 | 0.1 | 0.320571 | 0.225997i |

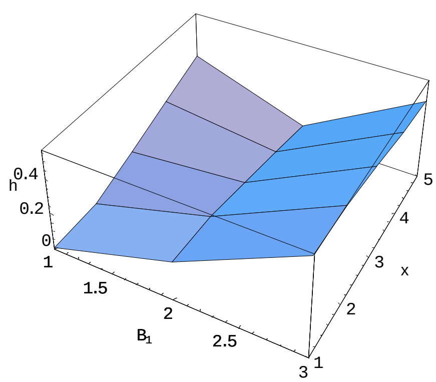


**Fig. 3. Surface representation of the effect of the quadruple mass moment for e=0.6, a=0.8 and $B_2$=-0.00001 of LP400-22**

# 4 Stability of Triangular Libration Points

To examine the linear stability of an infinitesimal body near the triangular point $L_4$ $(\xi_0, \eta_0)$ we displace it to a position $\xi = \xi_0 + \alpha, \eta = \eta_0 + \beta$, where $\alpha, \beta$ are small displacements. Substituting these values in the equations of motion (1) and taking only the linear terms, the variational equations of motion corresponding to the system are given as: -

$$\xi'' - 2\eta' = \alpha\Omega^o_{\xi\xi} + \beta\Omega^o_{\eta\eta}$$

$$\eta'' + 2\xi' = \alpha\Omega^o_{\eta\xi} + \beta\Omega^o_{\eta\eta}$$

The second order partial derivatives of $\Omega$ are represented by the subscripts, while the superscript 0 implies that the partial derivatives are to be evaluated at the libration point $L_4$ $(\xi_0, \eta_0)$.

Hence, the characteristics equation corresponding to the system is: -

$$\lambda^4 + \lambda^2 \, (4 - \Omega^o_{\eta\eta} - \Omega^o_{\xi\xi}) + \Omega^o_{\xi\xi}\Omega^o_{\eta\eta} - (\Omega^o_{\eta\xi})^2 = 0 \tag{13}$$

Neglecting second and higher order terms of $B_1, B_2, e^2$ and their products, the values of the partial derivatives at the triangular point (12) are obtained as

$$\Omega^o_{\xi\xi} = \frac{1}{(1 - e^2)^{½}} [\frac{3}{4a^{\frac{2}{3}}} + \frac{3}{4a^{\frac{2}{3}}} e^2 + B_1 \left\{\frac{3}{2a^{\frac{2}{3}}} + \frac{3\mu}{4a^{\frac{2}{3}}}\right\} + B_2 \left\{\frac{15}{16a^{\frac{2}{3}}} - \frac{45\mu}{16a^{⅔}} - \frac{15\mu}{16a^2} - \frac{15\mu}{4a^{\frac{4}{3}}}\right\}]$$

$$\Omega^o_{\eta\eta} = \frac{1}{(1 - e^2)^{½}} [3 - \frac{3}{4a^{\frac{2}{3}}} - \frac{3}{4a^{\frac{2}{3}}} e^2 + \frac{3}{4a^{\frac{2}{3}}} B_1 + B_2 \left\{- \frac{15}{8a^{\frac{4}{3}}} + \frac{15\mu}{16a^2} - \frac{15\mu}{4a^{\frac{4}{3}}} + \frac{15}{16a^{⅔}}\right\}]$$

$$\Omega^o_{\eta\xi} = \frac{\eta}{(1 - e^2)^{½}} [\frac{3}{2a^{\frac{2}{3}}} - \frac{3\mu}{2a^{\frac{2}{3}}} + e^2 \left\{\frac{3}{2a^{\frac{2}{3}}} - \frac{3\mu}{a^{⅔}}\right\} + B_1 \left\{- \frac{3\mu}{4a^{⅔}}\right\}$$
$$+ B_2 \{ \frac{15}{8a^{\frac{4}{3}}} - \frac{15}{8a^{\frac{2}{3}}} + \frac{15\mu}{4a^{⅔}} + \frac{15\mu}{16a^2} \}]$$

By substituting $a = 1 - \alpha$, simplifying, and neglecting product and higher order terms, we obtain,

$$\Omega^o_{\xi\xi} + \Omega^o_{\eta\eta} = 3 \, (1 + \frac{e^2}{2} - \frac{5\mu}{2} B_2)$$

$$(\Omega^o_{\xi\xi})(\Omega^o_{\eta\eta}) = [\frac{27}{16} + \frac{45^2}{16} e^2 + \frac{3}{4}\alpha + \frac{27\mu}{4} B_1 + \frac{45}{32} B_2 - \frac{45\mu}{4} B_2 ]$$

$$(\Omega^o_{\xi\eta})^2 = [\frac{27}{16} + \frac{27\mu^2}{4} - \frac{27\mu}{4} + \frac{45}{16} e^2 - \frac{45\mu}{16} e^2 + \frac{45\mu^2}{16} e^2 + \frac{3}{4}\alpha - 3\mu(1 - \mu)\alpha$$
$$+ B_1 \left\{- \frac{9}{2} - \frac{63\mu}{8} - \frac{9\mu^2}{8}\right\} + B_2 \left\{\frac{45}{32} + \frac{225\mu}{32} - \frac{315\mu^2}{16}\right\}$$

Substituting these values into equation (13) above and neglecting product and higher order terms, we get,

$$4(\lambda^2)^2 + 4(4 - 3\Psi_1)\lambda^2 + 27\mu(1 - \mu) + 4\Psi_2 = 0 \tag{14}$$

Where;

$$\Psi_1 = 1 + \frac{e^2}{2} + \mu A_2 - \frac{5\mu}{2} B_2$$

And

$$\Psi_2 = 3\mu(1-\mu)\alpha + 9\mu(1-\mu)B_1 + \frac{45\mu(1-\mu)}{4}e^2 - \frac{315\mu(1-\mu)}{16}B_2$$

Equation (14) is a quadratic equation in $\boldsymbol{\lambda^2}$, which yields;

$$\lambda^2 = \frac{-(4-3\Psi_1) \pm \{(4-3\Psi_1)^2 - [27\mu(1-\mu) + 4\Psi_2]\}^{½}}{2}$$

Its roots are

$$\lambda^2 = \frac{-b \pm \sqrt{\Delta}}{2} \tag{15}$$

where the discriminant

$$\Delta = (4-3\Psi_1)^2 - [27\mu(1-\mu) + 4\Psi_2] \tag{16}$$

From equation (16) above;

$$\Delta = (4-3\Psi_1)^2 - 27\mu + 27\mu^2 - 4\Psi_2] = 27\mu^2 + 12\mu^2\alpha + 45\mu^2e^2 + 36\mu^2B_1 - \frac{315\mu^2}{4}B_2 - 27\mu - 12\mu\alpha - 45\mu e^2 - 36\mu B_1 + \frac{375\mu}{4}B_2 + 1 - 3e^2 > 0 \tag{17}$$

For the stability of the libration points as given in equation (16) above, and equating the discriminant to zero i.e. $\Delta = 0$ and solving for μ, we obtain the critical mass parameter $\mu_c$ as:

$$\mu_c = \mu_o - \left[\frac{4}{27\sqrt{69}}\right]\alpha - \left[\frac{14}{9\sqrt{69}}\right]e^2 - \frac{1}{9}\left[1 + \frac{13}{\sqrt{69}}\right]B_1 - \frac{5}{18}\left[1 - \frac{25}{2\sqrt{69}}\right]B_2$$

Where,

$$\mu_o = \frac{1}{2}\left[1 - \sqrt{\frac{23}{27}}\right]$$

The value of the critical mass parameter to ten decimal places is: -

$$\mu_c = 0.0385208965 - 0.0178349412\alpha - 0.1872668826e^2 - 0.06277956556B_1 + 0.1402286564B_2 \tag{18}$$

Since $\Delta > 0$, in the interval $0 < \mu < \mu_c$, this implies that the roots of equation (15) are pure imaginary numbers, hence the triangular points are stable in this region. In the interval $\mu_c < \mu < ½$, $\Delta < 0$, the real parts of the two roots (15) are positive, therefore the triangular points are unstable. Now, if $\mu = \mu_c$, $\Delta = 0$, the roots in (15) are double roots, hence the instability of these points.

The triangular points are thus stable in the interval $0 < \mu < \mu_c$, where the critical mass parameter depends on the influence oblateness up to $J_4$, the semi-major axis and eccentricity of the orbits.

# 5 Conclusions

The positions and linear stability of the triangular libration points have been investigated in the ER3BP and are found to be affected by oblateness ($J_2$ & $J_4$), eccentricity (e) and semi-major axis (a) of the orbits. These effects on the locations are shown numerically (Tables 2-7) and graphically (Figs. 1-3). It is observed generally that for the binary systems NLTT 11748, LP400-22 and J1257+5428, the effect of increasing eccentricity is a shift away from the line joining the primaries, and reverse is the case with increase in the semi-major axis (Tables 3, 5 and 7). The triangular points also cease to exist as the semi-major axis approaches unity and in the quasi-parabolic cases. The quadruple and octupolar mass moment on the other hand, cause a shift towards the origin and away from the line joining the primaries respectively. Our results in the circular case confirm with [7,27] in the absence of radiation in their problem and with $J_4$=0 in ours. They also agree with those of [4,5] when the primary is spherical and secondary is non-luminous in the latter cases and [10] without considering oblateness of the bigger primary as well as neglecting the gravitational potential from the circular cluster of material points. Under the same conditions in the elliptic case when the octupolar mass moment $J_4$ is taken as zero, it verifies the results of [28]. The effects of the perturbations in oblateness ($J_2$ & $J_4$) and eccentricity of the orbits (18) is a reduction in the size of the region of stability. As such, they have destabilizing tendencies. The triangular points are stable in the interval $0 < \mu < \mu_c$. The mass ratios of the double white dwarf binaries use in this paper (Table 1) are outside this range and are as such unstable.

# Competing Interests

Authors have declared that no competing interests exist.

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