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On the Asymptotic Stability of Solutions for a Certain Non-autonomous Third-order Delay Differential Equation

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

In this paper, we shall establish sufficient conditions for the asymptotic stability of the zero solution for a certain nonlinear non-autonomous third-order delay differential equation of the following type

$$\dddot{x} + a(t)\ddot{x} + b(t)g(\dot{x}(t - r(t))) + c(t)h(x(t - r(t))) = 0.$$

By using a Lyapunov functional as a basic technique, we obtain a result which includes and improves some related results in literature. An example is given in the last section of this paper, to illustrate our main result of stability.

Keywords: Asymptotic stability; non-autonomous third-order delay differential equation; Lyapunov functionals.

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1 Introduction

It is well-known that the study of qualitative properties of solutions, in particular an investigation of their stability, is a very important problem in the theory and applications of differential equations. The stability of solutions of delay differential equations has been studied by a variety of authors over the years. We mention only a few of such books, for example, [1], [2], [3], [4], [5] and other references therein.

So far perhaps the most effective method to determine the stability behaviour of solutions of linear and nonlinear differential equations, with or without delay, is still the Lyapunov's second method.

The major advantage of this method is that stability in the large can be obtained without any prior knowledge of solutions. That is the method yields stability information directly without solving the equation.

Lyapunov functional is an interesting and fruitful technique to determine the stability behaviour of solutions of linear and nonlinear differential equations. This technique has gained increasing significance and has given impetus for modern development of stability theory of differential equations.

Besides it is worth-mentioning, that according to our observation, it can be seen some papers on the stability of solutions of third-order delay differential equations (see, for example, Abou-El-Ela et. al. [6], Ademola and Arawomo [7], Afuwape and Omeike [8], Bai and Guo [9], Omeike [10] and [11], Remili and Beldjerd [12] and [13], Remili and Oudjedi [14], Sadek[15] and [16], Shekhare et. al. [17], Tẹjumọla and Tchegnani [18], Tunç [19], [20], [21], [22], [23], [24] and [25], Zhu [26]) and references quoted therein.

In this work, we consider the nonlinear non-autonomous third-order delay differential equation of the following form

$$\dddot{x} + a(t)\ddot{x} + b(t)g(\dot{x}(t - r(t))) + c(t)h(x(t - r(t))) = 0, \tag{1.1}$$

where $0 \le r(t) \le \gamma$, γ is a positive constant which will be determined later; $a(t), b(t), c(t),$ $g(\dot{x})$ and $h(x)$ are real-valued functions continuous in their respective arguments; $g(0) = h(0) = 0$.

The dots indicate differentiation with respect to t and all solutions are assumed real.

Equation of the form (1.1), in which $a(t), b(t), c(t)$ and $r(t)$ are constants has been studied by several authors, namely: Zhu [26], Sadek [15] and other references therein.

But Abou-El-Ela, Sadek and Mahmoud [6] and Tunç [19] considered $a(t), b(t)$ and $c(t)$ as constants; $0 \le r(t) \le \gamma$, γ is a positive constant. To mention a few, they obtained criteria which ensure stability and boundedness of solutions.

On the other hand, for a kind of non-autonomous nonlinear third-order delay differential equations, the stability and boundedness results have been investigated only by a few researchers such as:

In 2005, Sadek [16] gave sufficient conditions for the asymptotic stability of the zero solution of third-order delay differential equation

$$\dddot{x} + a(t)\ddot{x} + b(t)\dot{x} + c(t)f(x(t - r)) = 0.$$

In 2008, Tunç [22] established some sufficient conditions for the asymptotic stability of the zero solution of nonlinear delay differential equation of third-order

$$\dddot{x} + a(t)\phi(x, \dot{x})\ddot{x} + b(t)\psi(x, \dot{x}) + c(t)h(x(t - r)) = 0.$$

In 2010, Tunç [24] obtained sufficient conditions for the stability of solutions of non-autonomous third-order differential equation with a deviating argument r as

$$\dddot{x} + a(t)\ddot{x} + b(t)g_1(\dot{x}(t-r)) + g_2(\dot{x}) + h(x(t-r)) = 0.$$

In 2010, Omeike [11] studied the stability of the same above equation on Sadek's [16] by another way.

Recently in 2013, Shekhar et. al. [17] investigated the conditions of stability of third-order non-autonomous nonlinear differential with delay

$$\dddot{x} + a(t)\ddot{x} + b(t)g(\dot{x}) + h(x(t-r)) = 0.$$

In this work, by constructing Lyapunov functional we obtain a new result of stability, which complement and extend the previously known results.

Remark 1.1. Clearly the equation discussed in Sadek [16] and in Omeike [11] is a special case of equation (1.1) when $r(t) = r$ and $g(\dot{x}) = \dot{x}$. Moreover, if $g(\dot{x}(t - r(t))) = g(\dot{x})$, $r(t) = r$ and $c(t) = 1$ reduces to the case studied by Shekhar et. al. [17].

2 Stability Results

In order to reach the main result of this paper, we shall give some basic information to the stability criteria for the general non-autonomous differential system with retarded argument

$$\dot{x} = f(t, x_t), \quad x_t = x(t+\theta), \quad -r \le \theta \le 0, \quad t \ge 0, \tag{2.1}$$

where $f : [0,\infty) \times \mathcal{C}_H \to^n$ is a continuous mapping, $f(t, 0) = 0$. We suppose that f takes closed bounded sets into bounded sets of ^{n}. Here $(\mathcal{C}, \|.\|)$ is the Banach space of continuous functions $\phi : [-r, 0] \to^n$ with supremum norm, $r > 0$; $\mathcal{C}_H$ is the open $H-$ball in $\mathcal{C}$; $\mathcal{C}_H := \{\phi \in \mathcal{C}([-r, 0],^n) : \|\phi\| < H\}$.

The following are the classical theorems on uniform stability and uniform asymptotic stability of (2.1). It goes back to Krasovskii [5].

Theorem 2.1. [27] *Let $V(t, \phi) : \mathcal{C}_H \to$ be a continuous functional satisfying a local Lipschitz condition and the functions $W_i(r)$, $(i = 1, 2)$ are wedges, satisfying*

(i) $W_1(|\phi(0)|) \le V(t, \phi) \le W_2(\|\phi\|)$ *and*

(ii) $\dot{V}_{(2.1)}(t, x_t) \le 0$.

Then the zero solution of (2.1) *is uniformly stable.*

Theorem 2.2. [27] *If there are a Lyapunov functional V for* (2.1) *and functions $W_i(r)$, $(i = 1, 2, 3)$ are wedges such that*

(i) $W_1(|\phi(0)|) \le V(t, \phi) \le W_2(\|\phi\|)$ *and*

(ii) $\dot{V}_{(2.1)}(t, x_t) \le -W_3(|x(t)|)$.

Then the zero solution of (2.1) *is uniformly asymptotically stable.*

The main objective of this paper is to prove the following theorem.

Theorem 2.3. *Suppose that $a(t), b(t)$ and $c(t)$ are continuously differentiable on $[0,\infty)$ and the following conditions are satisfied*

(i) $h(0) = 0$, $\frac{h(x)}{x} \ge \delta_0 > 0$ $(x \ne 0)$ *and* $h'(x) \le c_1$, *for all* x.

(ii) $g(0) = 0$, $\frac{g(y)}{y} \geq b > 0$ $(y \neq 0)$ *and* $g'(x) \leq c_2$, *for all* y.

(iii) $0 < \delta_1 \leq c(t) \leq b(t)$, $-L \leq b'(t) \leq c'(t) \leq 0$, *for* $t \geq 0$.

(iv) $0 \leq \Delta \leq a(t) \leq L$, *for* $t \geq 0$.

(v) $\frac{1}{2}a'(t) \leq \delta_2 \leq \delta_1(b - \alpha c_1)$, *for* $t \geq 0$.

(vi) $r(t) \leq \gamma$ *and* $r'(t) \leq \beta$, $0 < \beta < 1$.

(vii) $\int_0^\infty |c'(t)|dt < \infty$, $c'(t) \to 0$ *as* $t \to \infty$.

Then the zero solution of (1.1) *is uniformly asymptotically stable, provided that*

$$\gamma < min\left\{\frac{\delta_3(1-\beta)}{c_1L(1+\alpha) + L(c_1+c_2)(1-\beta)}, \frac{(\alpha\Delta - 1)(1-\beta)}{c_2L(1+\alpha) + L\alpha(c_1+c_2)(1-\beta)}\right\},$$

where

$$\delta_3 := \delta_1(b - \alpha c_1) - \delta_2 > 0.$$

The following Remark is important for the proof of the main result.

Remark 2.1. From (*iii*) it follows that $b(t)$ and $c(t)$ are non-decreasing functions on $[0, \infty)$. Thus, since they are continuous on this interval and bounded below by $\delta_1 > 0$, they are bounded on $[0, \infty)$ and the limit of each exists at $t \to \infty$.

Since L in (*iii*) and (*iv*) is an arbitrary selected bound, we can also assume that

$$\begin{aligned} 0 \leq \delta_1 \leq c(t) \leq b(t) \leq L, \\ \lim_{t\to\infty} c(t) = c_0, \quad \lim_{t\to\infty} b(t) = b_0, \\ \delta_1 \leq c_0 \leq b_0 \leq L. \end{aligned} \tag{2.2}$$

Proof of Theorem 2.3. We write equation (1.1) as the following equivalent system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -a(t)z - b(t)g(y) - c(t)h(x) \\ &\quad + b(t)\int_{t-r(t)}^{t} g'(y(s))z(s)ds + c(t)\int_{t-r(t)}^{t} h'(x(s))y(s)ds. \end{aligned} \tag{2.3}$$

Define the Lyapunov functional as

$$V(t, x_t, y_t, z_t) = e^{-\nu(t)}\, U(t, x_t, y_t, z_t), \tag{2.4}$$

where $\nu(t) = \int_0^t |c'(s)|ds$. It may be assumed that $\int_0^\infty |c'(t)|dt \leq N < \infty$ and

$$\begin{aligned} U(t, x_t, y_t, z_t) =& c(t)\int_0^x h(\xi)d\xi + \alpha c(t)h(x)y + \frac{1}{2}a(t)y^2 + \alpha b(t)\int_0^y g(\eta)d\eta + yz \\ &+ \frac{1}{2}\alpha z^2 + \lambda\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \mu\int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds, \end{aligned} \tag{2.5}$$

where $\alpha > 0$ is any number chosen such that

$$\frac{1}{\Delta} < \alpha < \frac{b}{c_1}, \tag{2.6}$$

and λ, μ are two positive constants, which will be determined later.

So that from (2.5) and (2.3), we find

$$\begin{aligned}\frac{dU}{dt} =& c'(t)\int_0^x h(\xi)d\xi + \alpha c'(t)h(x)y + \alpha c(t)h'(x)y^2 + \frac{1}{2}a'(t)y^2 \\ &+ \alpha b'(t)\int_0^y g(\eta)d\eta + z^2 - \alpha a(t)z^2 - b(t)g(y)y \\ &+ (y+\alpha z)\left\{b(t)\int_{t-r(t)}^t g'(y(s))z(s)ds + c(t)\int_{t-r(t)}^t h'(x(s))y(s)ds\right\} \\ &+ \lambda y^2 r(t) - \lambda(1-r'(t))\int_{t-r(t)}^t y^2(\theta)d\theta + \mu z^2 r(t) - \mu(1-r'(t))\int_{t-r(t)}^t z^2(\theta)d\theta.\end{aligned}$$

Since $h'(x) \le c_1$, $c(t) \le L$ by (2.2); and by using the inequality $2uv \le u^2 + v^2$, we have

$$\begin{aligned}c(t)(y+\alpha z)\int_{t-r(t)}^t h'(x(s))y(s)ds \le& \frac{1}{2}c_1 L r(t)y^2 + \frac{1}{2}c_1\alpha L r(t)z^2 \\ &+ \frac{1}{2}c_1 L(1+\alpha)\int_{t-r(t)}^t y^2(s)ds.\end{aligned}$$

Also since $g'(y) \le c_2$, $b(t) \le L$ by (2.2); and by using the inequality $2uv \le u^2 + v^2$, we obtain

$$\begin{aligned}b(t)(y+\alpha z)\int_{t-r(t)}^t g'(y(s))z(s)ds \le& \frac{1}{2}c_2 L r(t)y^2 + \frac{1}{2}c_2\alpha L r(t)z^2 \\ &+ \frac{1}{2}c_2 L(1+\alpha)\int_{t-r(t)}^t z^2(s)ds.\end{aligned}$$

Therefore we get

$$\begin{aligned}\frac{dU}{dt} \le& c'(t)\int_0^x h(\xi)d\xi + \alpha c'(t)h(x)y + \alpha c(t)h'(x)y^2 + \frac{1}{2}a'(t)y^2 + \alpha b'(t)\int_0^y g(\eta)d\eta \\ &+ z^2 - \alpha a(t)z^2 - b(t)g(y)y + \frac{1}{2}L(c_1+c_2)r(t)y^2 + \frac{1}{2}\alpha L(c_1+c_2)r(t)z^2 \\ &+ \frac{1}{2}c_1 L(1+\alpha)\int_{t-r(t)}^t y^2(s)ds + \frac{1}{2}c_2 L(1+\alpha)\int_{t-r(t)}^t z^2(s)ds \\ &+ \lambda y^2 r(t) - \lambda(1-r'(t))\int_{t-r(t)}^t y^2(\theta)d\theta + \mu z^2 r(t) - \mu(1-r'(t))\int_{t-r(t)}^t z^2(\theta)d\theta.\end{aligned}$$

Since $r(t) \le \gamma$ and $r'(t) \le \beta$, we find

$$\begin{aligned}\frac{dU}{dt} \le& c'(t)\int_0^x h(\xi)d\xi + \alpha c'(t)h(x)y + \alpha c(t)h'(x)y^2 + \frac{1}{2}a'(t)y^2 + \alpha b'(t)\int_0^y g(\eta)d\eta \\ &+ z^2 - \alpha a(t)z^2 - b(t)g(y)y + \frac{1}{2}L\gamma(c_1+c_2)y^2 + \frac{1}{2}\alpha L\gamma(c_1+c_2)z^2 \\ &+ \lambda y^2\gamma + \left\{\frac{1}{2}c_1 L(1+\alpha) - \lambda(1-\beta)\right\}\int_{t-r(t)}^t y^2(s)ds \\ &+ \mu z^2\gamma + \left\{\frac{1}{2}c_2 L(1+\alpha) - \mu(1-\beta)\right\}\int_{t-r(t)}^t z^2(s)ds.\end{aligned}$$

If we take

$$\lambda = \frac{c_1 L(1+\alpha)}{2(1-\beta)} > 0, \qquad \mu = \frac{c_2 L(1+\alpha)}{2(1-\beta)} > 0. \tag{2.7}$$

So that

$$\begin{aligned}\frac{dU}{dt} \leq & c'(t)\int_0^x h(\xi)d\xi + \alpha c'(t)h(x)y + \frac{1}{2}a'(t)y^2 + \alpha b'(t)\int_0^y g(\eta)d\eta \\ & - \left[c(t)\left\{\frac{b(t)}{c(t)}\frac{g(y)}{y}\right\}y^2 - \alpha c(t)h'(x)y^2 - \lambda\gamma y^2 - \frac{1}{2}L\gamma(c_1+c_2)y^2\right] \\ & - \frac{1}{2}\left\{2(\alpha a(t)-1) - L\alpha\gamma(c_1+c_2) - 2\mu\gamma\right\}z^2.\end{aligned}$$

From (i), (ii) and (2.2), we have

$$\begin{aligned}\frac{dU}{dt} \leq & c'(t)\int_0^x h(\xi)d\xi + \alpha b'(t)\int_0^y g(\eta)d\eta + \alpha c'(t)h(x)y + \frac{1}{2}a'(t)y^2 \\ & - \left\{\delta_1(b-\alpha c_1) - \frac{1}{2}L\gamma(c_1+c_2) - \lambda\gamma\right\}y^2 \\ & - \frac{1}{2}\left\{2(\alpha a(t)-1) - L\alpha\gamma(c_1+c_2) - 2\mu\gamma\right\}z^2.\end{aligned}$$

According to (iv), $\alpha a(t) \geq \alpha\Delta > 1$ by (2.6); thus

$$\begin{aligned}\frac{dU}{dt} \leq & c'(t)\int_0^x h(\xi)d\xi + \alpha b'(t)\int_0^y g(\eta)d\eta + \alpha c'(t)h(x)y + \frac{1}{2}a'(t)y^2 \\ & - \left\{\delta_1(b-\alpha c_1) - \frac{1}{2}L\gamma(c_1+c_2) - \lambda\gamma\right\}y^2 \\ & - \frac{1}{2}\left\{2(\alpha\Delta-1) - L\alpha\gamma(c_1+c_2) - 2\mu\gamma\right\}z^2.\end{aligned}$$

From (v), it follows that

$$\begin{aligned}\frac{d}{dt}U(t,x_t,y_t,z_t) \leq & c'(t)\int_0^x h(\xi)d\xi + \alpha b'(t)\int_0^y g(\eta)d\eta + \alpha c'(t)h(x)y \\ & + \left\{\delta_2 - \delta_1(b-\alpha c_1) + \frac{1}{2}L\gamma(c_1+c_2) + \lambda\gamma\right\}y^2 \\ & - \frac{1}{2}\left\{2(\alpha\Delta-1) - L\alpha\gamma(c_1+c_2) - 2\mu\gamma\right\}z^2.\end{aligned}$$

If we let $\delta_3 := \delta_1(b-\alpha c_1) - \delta_2 > 0$, then from (2.7) we find

$$\begin{aligned}\frac{d}{dt}U(t,x_t,y_t,z_t) \leq & c'(t)\int_0^x h(\xi)d\xi + \alpha b'(t)\int_0^y g(\eta)d\eta + \alpha c'(t)h(x)y \\ & - \left\{\delta_3 - \frac{1}{2}L\gamma(c_1+c_2) - \frac{c_1L(1+\alpha)}{2(1-\beta)}\gamma\right\}y^2 \\ & - \left\{(\alpha\Delta-1) - \frac{1}{2}L\alpha\gamma(c_1+c_2) - \frac{c_2L(1+\alpha)}{2(1-\beta)}\gamma\right\}z^2.\end{aligned} \tag{2.8}$$

Next we show that

$$c'(t)\int_0^x h(\xi)d\xi + \alpha b'(t)\int_0^y g(\eta)d\eta + \alpha c'(t)h(x)y \leq 0, \text{ for all } x, y \text{ and } t \geq 0.$$

From (iii), $-L \leq b'(t) \leq c'(t) \leq 0$ for $t \geq 0$, if $c'(t) = 0$ then

$$c'(t)\int_0^x h(\xi)d\xi + \alpha b'(t)\int_0^y g(\eta)d\eta + \alpha c'(t)h(x)y = \alpha b'(t)\int_0^y g(\eta)d\eta \leq 0,$$

since $b'(t) \le 0$ and $\int_0^y g(\eta)d\eta \ge 0$.

For those t's such that $c'(t) < 0$ and $\frac{b'(t)}{c'(t)} < 1$; by (iii) we have

$$\begin{aligned} &c'(t)\int_0^x h(\xi)d\xi + \alpha b'(t)\int_0^y g(\eta)d\eta + \alpha c'(t)h(x)y \\ &\le c'(t)\Big\{\int_0^x h(\xi)d\xi + \alpha\int_0^y g(\eta)d\eta + \alpha h(x)y\Big\}. \end{aligned}$$

Since $h'(x) \le c_1$ and $\frac{g(y)}{y} \ge b > 0$ implies that $\int_0^y g(\eta)d\eta \ge \frac{1}{2}by^2$, therefore we find

$$\begin{aligned} &c'(t)\int_0^x h(\xi)d\xi + \alpha b'(t)\int_0^y g(\eta)d\eta + \alpha c'(t)h(x)y \\ &\le c'(t)\Big\{\frac{1}{2}\frac{\alpha}{b}(by + h(x))^2 + \int_0^x (1 - \frac{\alpha}{b}h'(\xi))h(\xi)d\xi\Big\} \\ &\le c'(t)\int_0^x (1 - \frac{\alpha c_1}{b})h(\xi)d\xi \le c'(t)\delta_4\int_0^x h(\xi)d\xi \le 0, \end{aligned}$$

where $\delta_4 \equiv 1 - \frac{\alpha c_1}{b} > 1 - \frac{(\frac{b}{c_1}).c_1}{b} = 0$, for all x, y and t.

Thus we can write (2.8) as the following

$$\begin{aligned} \frac{d}{dt}U(t, x_t, y_t, z_t) \le &- \Big\{\delta_3 - \frac{1}{2}L\gamma(c_1 + c_2) - \frac{c_1L(1+\alpha)}{2(1-\beta)}\gamma\Big\}y^2 \\ &- \Big\{(\alpha\Delta - 1) - \frac{1}{2}L\alpha\gamma(c_1 + c_2) - \frac{c_2L(1+\alpha)}{2(1-\beta)}\gamma\Big\}z^2. \end{aligned}$$

If we choose

$$\gamma < min\Big\{\frac{\delta_3(1-\beta)}{c_1L(1+\alpha) + L(c_1+c_2)(1-\beta)}, \frac{(\alpha\Delta - 1)(1-\beta)}{c_2L(1+\alpha) + L\alpha(c_1+c_2)(1-\beta)}\Big\},$$

then we have

$$\frac{d}{dt}U(t, x_t, y_t, z_t) \le -k_1(y^2 + z^2), \text{ for some } k_1 > 0. \tag{2.9}$$

Since the integrals $\lambda\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds$ and $\mu\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds$ are non-negative, then from (2.5) we have

$$U \ge c(t)\Big\{\int_0^x h(\xi)d\xi + \alpha\frac{b(t)}{c(t)}\int_0^y g(\eta)d\eta + \alpha h(x)y\Big\} + \frac{1}{2}a(t)y^2 + yz + \frac{1}{2}\alpha z^2.$$

From the conditions $\frac{b(t)}{c(t)} \ge 1$, $c(t) \ge \delta_1 \ge 0$; by (iii), and $\frac{g(y)}{y} \ge b > 0$ implies that $\int_0^y g(\eta)d\eta \ge \frac{1}{2}by^2$, then we obtain

$$\begin{aligned} U(t, x_t, y_t, z_t) \ge &\delta_1\Big\{\int_0^x h(\xi)d\xi + \frac{1}{2}\frac{\alpha}{b}(by + h(x))^2 - \frac{1}{2}\frac{\alpha}{b}h^2(x)\Big\} \\ &+ \frac{1}{2}a(t)\{y + \frac{z}{a(t)}\}^2 + \frac{1}{a(t)}(\alpha a(t) - 1)z^2. \end{aligned}$$

According to (iv) and (2.6), clearly $\alpha a(t) - 1$ is positive. Thus there exist a positive constant δ_5

such that

$$\begin{aligned}U(t,x_t,y_t,z_t) &\geq \delta_1\left[\int_0^x\left\{1-\frac{\alpha}{b}h'(\xi)\right\}h(\xi)d\xi+\frac{1}{2}\frac{\alpha}{b}(by+h(x))^2\right]+\frac{1}{2}\delta_5(y^2+z^2)\\ &\geq \delta_1\int_0^x\left\{1-\frac{\alpha}{b}h'(\xi)\right\}h(\xi)d\xi+\frac{1}{2}\delta_5(y^2+z^2)\\ &\geq \delta_1\delta_4\int_0^x h(\xi)d\xi+\frac{1}{2}\delta_5(y^2+z^2).\end{aligned}$$

Since $\frac{h(x)}{x}\geq\delta_0>0$, then we have

$$U(t,x_t,y_t,z_t)\geq\frac{1}{2}\delta_0\delta_1\delta_4x^2+\frac{1}{2}\delta_5(y^2+z^2).$$

Then there exists a positive constant k_2 such that

$$U(t,x_t,y_t,z_t)\geq k_2(x^2+y^2+z^2), \tag{2.10}$$

where $k_2 := min\{\frac{\delta_0\delta_1\delta_4}{2},\frac{\delta_5}{2}\}$.

Therefore we can find a continuous function $W_1(|\phi(0)|)$ with $W_1(|\phi(0)|)>0$ and $W_1(|\phi(0)|)\leq V(t,\phi)$.

Now we shall prove that there exists a continuous function $W_2(\|\phi\|)$ which satisfies the inequality $V(t,\phi)\leq W_2(\|\phi\|)$.

Since $h'(x)\leq c_1$, $g'(y)\leq c_2$ and $h(0)=g(0)=0$, then by using the mean-value theorem we find $h(x)\leq c_1x$ and $g(y)\leq c_2y$. Therefore we obtain

$$\begin{aligned}U(t,x_t,y_t,z_t)\leq&\frac{1}{2}Lc_1x^2+\alpha Lc_1|xy|+\frac{1}{2}Ly^2+\frac{1}{2}\alpha Lc_2y^2+|yz|+\frac{1}{2}\alpha z^2\\ &+\lambda\int_{t-r(t)}^t(\theta-t+r(t))y^2(\theta)d\theta+\mu\int_{t-r(t)}^t(\theta-t+r(t))z^2(\theta)d\theta.\end{aligned}$$

From (vi) and since $|uv|\leq\frac{1}{2}(u^2+v^2)$, then we get

$$\begin{aligned}U(t,x_t,y_t,z_t)&\leq\frac{1}{2}Lc_1x^2+\frac{1}{2}\alpha Lc_1(x^2+y^2)+\frac{1}{2}Ly^2+\frac{1}{2}\alpha Lc_2y^2+\frac{1}{2}(y^2+z^2)+\frac{1}{2}\alpha z^2\\ &\quad+\frac{1}{2}\lambda r^2(t)\|y\|^2+\frac{1}{2}\mu r^2(t)\|z\|^2\\ &\leq\frac{1}{2}Lc_1(1+\alpha)\|x\|^2+\frac{1}{2}\{\alpha L(c_1+c_2)+1+L+\lambda\gamma^2\}\|y\|^2\\ &\quad+\frac{1}{2}(1+\alpha+\mu\gamma^2)\|z\|^2.\end{aligned}$$

Therefore there exists a positive constant k_3 such that

$$U(t,x_t,y_t,z_t)\leq k_3(x^2+y^2+z^2), \tag{2.11}$$

where $k_3 := min\left\{\frac{Lc_1(1+\alpha)}{2},\frac{\alpha L(c_1+c_2)+1+L+\lambda\gamma^2}{2},\frac{1+\alpha+\mu\gamma^2}{2}\right\}$.

From (2.4) we have

$$\frac{d}{dt}V(t,x_t,y_t,z_t)=e^{-\nu(t)}\left\{\frac{d}{dt}U(t,x_t,y_t,z_t)-|c'(t)|U(t,x_t,y_t,z_t)\right\}.$$

By using the inequalities (2.9), (2.10) and the fact that $|c'(t)| \geq 0$, we obtain

$$\frac{dU}{dt} - |c'(t)|U \leq -k_1(y^2 + z^2) - k_2(x^2 + y^2 + z^2),$$

therefore, if

$$\gamma < min\left\{\frac{\delta_3(1-\beta)}{c_1L(1+\alpha) + L(c_1+c_2)(1-\beta)}, \frac{(\alpha\Delta - 1)(1-\beta)}{c_2L(1+\alpha) + L\alpha(c_1+c_2)(1-\beta)}\right\},$$

then we have

$$\begin{aligned} \frac{d}{dt}V(t, x_t, y_t, z_t) &\leq -k_4\ e^{-\nu(t)}(x^2 + y^2 + z^2), \quad \text{for some } \ k_4 > 0 \\ &\leq -W_3(|x(t)|). \end{aligned} \tag{2.12}$$

Therefore from (2.10), (2.11) and (2.12) the Lyapunov functional $V(t, x_t, y_t, z_t)$ satisfies all the conditions of Theorem 2.2, so that the zero solution of (1.1) is uniformly asymptotically stable.

Thus the proof of Theorem 2.3 is now complete.

3 Example

In this section, we give an example to illustrate the main stability result.

We consider the following third-order nonlinear non-autonomous delay differential equation

$$\begin{aligned} &\dddot{x} + (\frac{1}{4}\sin t + \frac{5}{4})\ddot{x} + (1 + \frac{1}{t^2+2})\{2\dot{x}(t - r(t)) + \sin \dot{x}(t - r(t))\} \\ &+ \frac{1}{28}(\frac{1}{4} + \frac{1}{t^2+3})x(t - r(t)) = 0. \end{aligned} \tag{3.1}$$

This equation can be stated as the following equivalent system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -(\frac{1}{4}\sin t + \frac{5}{4})z - (1 + \frac{1}{t^2+2})(2y + \sin y) \\ &+ (1 + \frac{1}{t^2+2})\int_{t-r(t)}^{t}\{2 + \cos y(s)\}z(s)ds \\ &- \frac{1}{28}(\frac{1}{4} + \frac{1}{t^2+3})x + \frac{1}{28}(\frac{1}{4} + \frac{1}{t^2+3})\int_{t-r(t)}^{t} y(s)ds. \end{aligned} \tag{3.2}$$

So we have

$$0 < \Delta = \frac{1}{4} \leq a(t) = \frac{1}{4}\sin t + \frac{5}{4} \leq \frac{3}{2} = L, \ \ \frac{1}{2}a'(t) = \frac{1}{8}\cos t \leq \frac{1}{8} = \delta_2, \text{ for all } \ t \geq 0,$$

$$0 < 1 \leq b(t) = 1 + \frac{1}{t^2+2} \leq \frac{3}{2}, \ \ \frac{-3}{2} < b'(t) = \frac{-2t}{(t^2+2)^2} \leq 0, \text{ for all } \ t \geq 0,$$

$$0 < \delta_1 = \frac{1}{4} \leq c(t) = \frac{1}{4} + \frac{1}{t^2+3} \leq \frac{7}{12}, \ \ \frac{-3}{2} < c'(t) = \frac{-2t}{(t^2+3)^2} \leq 0.$$

Then we can note that

$$0 < \frac{1}{4} = \delta_1 \le c(t) \le b(t) \le L = \frac{3}{2},$$
$$\lim_{t\to\infty} c(t) = c_0 = \frac{1}{4}, \quad \lim_{t\to\infty} b(t) = b_0 = 1, \quad \delta_1 \le c_0 \le b_0 \le L.$$

Now from (3.1) and (3.2) we obtain

$$g(y) = 2y + \sin y, \; g(0) = 0, \; \frac{g(y)}{y} = 2 + \frac{\sin y}{y} \ge 1 = b > 0 \;\; (y \ne 0),$$
$$g'(y) = 2 + \cos y, \;\; g'(y) \le 3 = c_2, \;\; \text{for all } y.$$

$$h(x) = \frac{1}{28}x, \; h(0) = 0, \; \frac{h(x)}{x} = \frac{1}{28} = \delta_0 > 0, \; h'(x) = \frac{1}{28} < \frac{1}{14}, \;\; \text{for all } x.$$

If we let $\alpha = 6$, then we get

$$\delta_1(b - \alpha c_1) = \frac{1}{7} \ge \delta_2 = \frac{1}{8}, \;\; \text{for all } t \ge 0, \; \text{then} \;\; \delta_3 := \delta_1(b - \alpha c_1) - \delta_2 = \frac{1}{56} > 0.$$

From the above definition of $c(t)$, we have

$$\int_0^\infty |c'(t)|dt = \frac{1}{3} < \infty, \;\; c'(t) \to 0 \quad as \quad t \to \infty.$$

Then all the assumptions of Theorem 2.3, (2.2) and (2.6) are satisfied, we can conclude using Theorem 2.3 that the zero solution of (3.1) is uniformly asymptotically stable.

4 Conclusion

The problem of the asymptotic stability of delay differential equations is very important in the theory and applications of differential equations. In the present paper, sufficient conditions were obtained for the asymptotic stability of the zero solution for a certain third-order nonlinear non-autonomous differential equation with the variable delay. By using a Lyapunov direct method as a basic technique, a Lyapunov functional was defined and used to obtain our results. The obtained results are new and extend existing results of stability in the literature on deterministic system of delay differential equation.

Competing Interests

Author has declared that no competing interests exist.

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