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On the Convergence of Gauss-type Proximal Point Method for Smooth Generalized Equations

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Authors' contributions

This work was carried out in collaboration between both authors. Author MAA gathered the initial data, helped to literature searching, managed the analysis of the study and drafting the article. Author MHR designed the study, created the idea to develop a mathematical algorithm and designed a methodology for programming and interpreted the results. Both authors read and approved the final manuscript.

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Abstract

Let X and Y be Banach spaces and Ω be an open subset of X. Let $f : X \to Y$ be a Fréchet differentiable function on Ω and $F : X \rightrightarrows 2^Y$ be a set valued mapping with closed graph. We deal with smooth generalized equations which is defined by the sum of Fréchet differentiable function and a set valued mapping. Under some sufficient conditions, a Gauss-type proximal point algorithm (G-PPA) is introduced and studied for solving generalized equations of the form $0 \in f(x) + F(x)$. Indeed, when F is metrically regular we analyze semi-local and local convergence of the G-PPA. Furthermore, we give a numerical example to justify the convergence results of the G-PPA.

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1 Introduction

Let X and Y be Banach spaces. We are involved with the problem of seeking a point $x \in \Omega \subseteq X$ satisfying

$$0 \in f(x) + F(x), \tag{1.1}$$

where $f : X \to Y$ is a Fréchet differentiable function and $F : X \rightrightarrows 2^Y$ is a set valued mapping with closed graph. Robinson [1, 2], introduced the generalized equation (1.1) for $f = 0$, as a general mechanism for describing, analyzing, and solving different problems in a unified way. Such kind of problems have been reviewed broadly. Various examples are system of inequalities, variational inequalities, linear and nonlinear complementary problems, system of nonlinear equations, equilibrium problems, etc.; see in [1, 2, 3].

It is clarify that when $F = \{0\}$, (1.1) is an equation. When F is the normal cone to a convex and closed set in X, (1.1) performs variational inequalities. When F is positive orthand in $\mathbb{R}^n$, (1.1) is a system of inequalities.

Different iterative methods have been presented for solving generalized equations such as Newton-type method, proximal point method, etc.; see in [4, 5, 6, 7]. The proximal point algorithm (PPA) is one of the most useful method for solving (1.1) in the case $f = 0$ and $Y = X$ a Hilbert space. About the root of PPA can be known in the works of Martinet [8] for variational inequalities. This PPA has been further polished and spread out in [3, 7, 9] to a more general framework, including convex programs, convex-concave saddle point problems and variational inequality problems. Rockafellar [7] earnestly analyzed the PPA in the general structure of maximal monotone inclusions.

Let $D(\lambda_k, x)$ denotes the subset of X for all $x \in X$ and for some sequence of positive numbers λ_k, which is characterized as follows:

$$D(\lambda_k, x) := \Big\{ d \in X : 0 \in \lambda_k d + f(x + d) + F(x + d) \Big\}. \tag{1.2}$$

Dontchev and Rockafellar [10] planned the following proximal point algorithm for solving (1.1):

Algorithm 1 (PPA)

Step 1. Let $x_0 \in X$, $\lambda > 0$ and put $k := 0$.
Step 2. If $0 \in D(\lambda_k, x_k)$, then stop; otherwise, go to Step 3.
Step 3. Put $\{\lambda_k\} \subseteq (0, \lambda)$ and if $0 \notin D(\lambda_k, x_k)$, choose d_k such that $d_k \in D(\lambda_k, x_k)$.

Step 4. Write $x_{k+1} := x_k + d_k$.
Step 5. Set k by $k + 1$ and go to Step 2.

Note that, for a starting point near to a solution, the sequences generated by Algorithm 1 are not uniquely defined and not every sequence is convergent. Under certain conditions, Dontchev and Rockafellar [10, Chapter 6] showed that there exists one sequence $\{x_n\}$ generated by Algorithm 1, which is linearly convergent to the solution. Hence, from the aspect of mathematical estimations, this type of methods are not agreeable in mathematical utilizations. This barrier inspire us to nominate a method "so called" Gauss-type proximal point algorithm (G-PPA). The difference between the Algorithm 1 and our proposed Algorithm 2 is that the G-PPA generates sequences, whose every sequence is convergent, but this does not happen for the Algorithm 1.

Algorithm 2 (G-PPA)

Step 1. Let $\eta \geq 1$, $x_0 \in X$, $\lambda > 0$ and put $k := 0$.
Step 2. If $0 \in D(\lambda_k, x_k)$, then stop; otherwise, go to Step 3.
Step 3. Put $\{\lambda_k\} \subseteq (0, \lambda)$ and if $0 \notin D(\lambda_k, x_k)$, choose d_k such that $d_k \in D(\lambda_k, x_k)$ and $\|d_k\| \leq \eta$ dist $(0, D(\lambda_k, x_k))$.
Step 4. Write $x_{k+1} := x_k + d_k$.
Step 5. Set k by $k + 1$ and go to Step 2.

We detect from the Algorithm 2, that

(i) if $\eta = 1$ and $D(\lambda_k, x_k)$ is singleton, Algorithm 2 matches with the Algorithm 1. For solving the generalized equation problem (1.1), Dontchev and Rockafellar [10, Chapter 6] established only the local convergence result. On the other hand, we have established both semilocal and local convergence results for solving (1.1).

(ii) if $\lambda_k u = g_k(u)$ a sequence of Lipschitz continuous functions, F is the normal cone mapping and $Y = X^*$ a dual Banach space of X, Algorithm 2 is identical to the Gauss-type proximal point method for variational inequalities, which has been introduced by Rashid [3]. In this case our Theorem 3.1 is identical with the result given by Rashid [3, Theorem 3.1].

(iii) if $f = 0$, and $Y = X$ a Banach space, Algorithm 2 is equivalent to the Gauss-type proximal point method, which have been introduced by Rashid et al. [11].

(iv) if $\lambda_k u = g_k(u)$ a sequence of Lipschitz continuous functions and $f = 0$, Algorithm 2 is comparable to the general version of Gauss-type proximal point algorithm, which have been introduced by Alom et al. [4].

There have been investigated many effective works on semi-local analysis for some special cases such as Newton method for nonlinear least square problems (cf. [5]), the extended Newton-type method for solving variational inclusions (cf. [12]) and the Gauss-Newton method for convex inclusion problems (cf. [13]). For seeking the solution of (1.1), Rashid et al. [6] introduced the Gauss-Newton type method and achieved the semi-local and local convergence results. In his sequential paper [3], Rashid introduced the Gauss-type proximal point method for finding the solution of variational inequality problem and obtained the semi-local and local convergence results. In recent time, Alom et al. [4] have been presented the general version of Gauss-type proximal point algorithm for solving (1.1) in the case $f = 0$ and analyzed the semi-local and local convergence results. To the best of our knowledge, there is no study on semi-local analysis for solving (1.1) by using the Gauss-type proximal point method. Thus, we conclude that the contributions, presented in this study, seem new.

In this study, our ambition is to evaluate the semi-local convergence of the G-PPA defined by Algorithm 2. The vital apparatus in our study are the metric regularity property, which was introduced by Dontchev and Rockafellar [14], and Lipschitz-like property for set-valued mappings, whose concept was introduced by Aubin [15, 16]. Our fundamental results are the convergence principle, entrenched in section 3, which, based on the information around the initial point, provide some sufficient conditions assure the convergence to a solution of any sequence generated by Algorithm 2. As a consequence, local convergence result for the G-PPA is achieved.

The content of this paper is arranged as follows: In section 2, we recall some significant notations, concepts, some preliminary results and also recall a fixed point theorem which has been proved by Dontchev and Hager (cf. [17]). This fixed-point theorem is the vital mechanism to prove the existence of any sequence generated by Algorithm 2. In section 3, we consider the G-PPA, which is introduced in this section, as well as the concept of metric regularity property and the Lipschitz-like property for set valued mappings to show the existence and the convergence of the sequence generated by Algorithm 2. To verify the convergence results of the G-PPA, we give a numerical example in section 4. In the last section, we give a summary of the main results obtained in the paper.

2 Notations and Preliminary Results

In the whole section, let X and Y be Banach spaces and let F be a set valued mapping from X into the subsets of Y, defined by $F : X \rightrightarrows 2^Y$. The graph of F is defined by the set $\text{gph}F := \{(x, y) \in X \times Y : y \in F(x)\}$, the domain of F is defined by $\text{dom}F := \{x \in X : F(x) \neq \emptyset\}$ and the inverse of F is defined by $F^{-1}(y) := \{x \in X : y \in F(x)\}$. By $\mathbb{B}_r(x)$, we denote the closed ball centered at x with radius r.

All the norms are denoted by $\|\cdot\|$. The distance from a point x to a set B is defined by $\text{dist}(x, B) := \inf\{\|x - a\| : a \in B\}$ for each $x \in X$, while the excess from a set E to the set B is defined by $e(E, B) := \sup\{\text{dist}(x, B) : x \in E\}$.

The concept in the following definition of metric regularity for a set valued mapping is taken from [[11]], and has been studied extensively; see for examples [9, 10, 18], and the references therein.

Definition 2.1. *Let $F : X \rightrightarrows 2^Y$ be a set-valued mapping and $(\bar{x}, \bar{y}) \in \text{gph}F$. Let $r_{\bar{x}} > 0$, $r_{\bar{y}} > 0$ and $\kappa > 0$. Then F is said to be*

(i) *metrically regular at $(\bar{x}, \bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant κ if for all $x \in \mathbb{B}_{r_{\bar{x}}}(\bar{x}), y \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$*

$$\text{dist}\big(x, F^{-1}(y)\big) \leq \kappa \ \text{dist}\big(y, F(x)\big).$$

(ii) *metrically regular at $(\bar{x}, \bar{y})$ if there exist constants $r'_{\bar{x}} > 0$, $r'_{\bar{y}} > 0$ and $\kappa' > 0$ such that F is metrically regular at $(\bar{x}, \bar{y})$ on $\mathbb{B}_{r'_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r'_{\bar{y}}}(\bar{y})$ with constant κ'.*

From [3], we recall the definition of Lipschitz-like continuity for set-valued mappings. This concept was introduced by Aubin [16] and has been studied extensively; see for examples [11, 14, 18] and the references therein.

Definition 2.2. *Let $\Gamma : Y \rightrightarrows 2^X$ be a set-valued mapping and let $(\bar{y}, \bar{x}) \in \text{gph}\Gamma$. Let $r_{\bar{x}} > 0$, $r_{\bar{y}} > 0$ and $M > 0$. Then Γ is said to be Lipschitz-like at $(\bar{y}, \bar{x})$ on $\mathbb{B}_{r_{\bar{y}}}(\bar{y}) \times \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with constant M if for any $y_1, y_2 \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$, the following inequality hold:*

$$e(\Gamma(y_1) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), \Gamma(y_2)) \leq M\|y_1 - y_2\|.$$

The equivalence relation between metric regularity of a mapping F and the Lipschitz-like continuity of the inverse F^{-1}, which can be seen in [9, 11], is given as follows:

Lemma 2.1. *Let $F : X \rightrightarrows 2^Y$ be a set valued mapping and $(\bar{x}, \bar{y}) \in \text{gph}F$. Let $r_{\bar{x}} > 0$, $r_{\bar{y}} > 0$ and $\kappa > 0$. Then F is metrically regular at $(\bar{x}, \bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant κ if and only if its inverse $F^{-1} : Y \rightrightarrows 2^X$ is Lipschitz-like at $(\bar{y}, \bar{x})$ on $\mathbb{B}_{r_{\bar{y}}}(\bar{y}) \times \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with constant κ, that is, for all $y, y' \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$,*

$$e(F^{-1}(y) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), F^{-1}(y')) \leq \kappa\|y - y'\|.$$

We finish this section with the following lemma. This lemma is known as Banach fixed point lemma which has been proved by Dontchev and Hagger in [17].

Lemma 2.2. *Let $\psi : X \rightrightarrows 2^X$ be a set-valued mapping. Let $\eta_0 \in X$, $r \in (0, \infty)$ and $\alpha \in (0, 1)$ be such that*

$$\operatorname{dist}(\eta_0, \psi(\eta_0)) < r(1-\alpha) \tag{2.1}$$

and for any $x_1, x_2 \in \mathbb{B}_r(\eta_0)$,

$$e(\psi(x_1) \cap \mathbb{B}_r(\eta_0), \psi(x_2)) \le \alpha \|x_1 - x_2\|. \tag{2.2}$$

Then ψ has a fixed point in $\mathbb{B}_r(\eta_0)$, that is, there exists $x \in \mathbb{B}_r(\eta_0)$ such that $x \in \psi(x)$. If ψ is single-valued, then there exists $x \in \mathbb{B}_r(\eta_0)$ such that $x = \psi(x)$.

3 Convergence Analysis of the G-PPA

Suppose X and Y are Banach spaces. Let $f : X \to Y$ be a single valued function, which is Fréchet differentiable on $\Omega \subseteq X$, and let $F : X \rightrightarrows 2^Y$ be a set valued mapping with closed graph. Let $r_{\bar{x}} > 0$, $r_{\bar{y}} > 0$, $\nu > 0$ and $\kappa > 0$ be such that $\nu\kappa < 1$. We define

$$r^* := \max\Big\{\frac{2r_{\bar{x}} + \kappa r_{\bar{y}}}{1-\nu\kappa}, \frac{2\nu r_{\bar{x}} + r_{\bar{y}}}{1-\nu\kappa}\Big\}. \tag{3.1}$$

From (3.1), it is obvious that $r_{\bar{x}} < r^*$ and $r_{\bar{y}} < r^*$.

To establish our main result, we need the following lemma:

Lemma 3.1. *Let $F : X \rightrightarrows 2^Y$ be a set valued mapping which has locally closed graph at $(\bar{x}, \bar{y}) \in$ gph$F$. Let r^* be defined by (3.1). Let F be metrically regular at $(\bar{x}, \bar{y})$ on $\mathbb{B}_{r^*}(\bar{x}) \times \mathbb{B}_{r^*}(\bar{y})$ with constant κ. Let $f : X \to Y$ be Lipschitz continuous on $\mathbb{B}_{r^*}(\bar{x})$ with Lipschitz constant ν and $f(\bar{x}) = 0$. Then the mapping $f + F$ is metrically regular at $(\bar{x}, \bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant $\dfrac{\kappa}{1-\nu\kappa}$.*

Proof. According to our assumption on F, we obtain

$$\operatorname{dist}\Big(x, F^{-1}(y)\Big) \le \kappa \operatorname{dist}\Big(y, F(x)\Big) \quad \text{for all } x \in \mathbb{B}_{r^*}(\bar{x}),\ y \in \mathbb{B}_{r^*}(\bar{y}).$$

For all $x \in \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ and $y \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$, we will show that

$$\operatorname{dist}\Big(x, (f+F)^{-1}(y)\Big) \le \frac{\kappa}{1-\nu\kappa} \operatorname{dist}\Big(y, (f+F)(x)\Big).$$

To complete this, we will proceed by induction on k and verify that there exists a sequence $\{x_k\} \subseteq \mathbb{B}_{r^*}(\bar{x})$, with $x_0 = x$, such that, for $k = 0, 1, 2, ...,$ satisfies the following assertions:

$$x_{k+1} \in F^{-1}(y - f(x_k)) \tag{3.2}$$

and

$$\|x_{k+1} - x_k\| \le (\nu\kappa)^k \|x_1 - x\|. \tag{3.3}$$

It is obvious that (3.3) is true for $k = 0$. From the second condition in (3.1), we get $2\nu r_{\bar{x}} + r_{\bar{y}} \le r^*(1-\nu\kappa)$ and since $\nu\kappa < 1$, so $(1-\nu\kappa)$ is positive, and hence $2\nu r_{\bar{x}} + r_{\bar{y}} \le r^*$. This implies that $\nu r_{\bar{x}} + r_{\bar{y}} \le r^*$. Thus, for all $x \in \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ and $y \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$, we have

$$\begin{aligned} \|(y - f(x)) - \bar{y}\| &= \|y - \bar{y} + f(\bar{x}) - f(x)\| \le \|f(x) - f(\bar{x})\| + \|y - \bar{y}\| \\ &\le \nu\|x - \bar{x}\| + \|y - \bar{y}\| \le \nu r_{\bar{x}} + r_{\bar{y}} \le r^*. \end{aligned} \tag{3.4}$$

This implies that $y - f(x) \in \mathbb{B}_{r^*}(\bar{y})$. Since F has locally closed graph, there exists $x_1 \in F^{-1}(y - f(x))$ with $x_0 = x$ and it shows that (3.2) is true for $k = 0$. Again, since F is metrically regular, we obtain

$$\|x_1 - x\| \le \operatorname{dist}\Big(x, F^{-1}(y - f(x))\Big) \le \kappa \operatorname{dist}\Big(y, (f+F)(x)\Big). \tag{3.5}$$

Also,

$$\begin{aligned}\|x_1 - x\| &= \|x_1 - \bar{x} + \bar{x} - x\| \leq \|x - \bar{x}\| + \|\bar{x} - x_1\| \leq r_{\bar{x}} + \text{dist}\Big(\bar{x}, F^{-1}(y - f(x))\Big) \\ &\leq r_{\bar{x}} + \kappa \ \text{dist}\Big(y - f(x), F(\bar{x})\Big) \leq r_{\bar{x}} + \kappa\|y - \bar{y}\| + \kappa\|f(x) - f(\bar{x})\| \\ &\leq r_{\bar{x}} + \kappa r_{\bar{y}} + \nu\kappa r_{\bar{x}} = (1 + \nu\kappa)r_{\bar{x}} + \kappa r_{\bar{y}}. \end{aligned} \tag{3.6}$$

Hence

$$\|x_1 - \bar{x}\| \leq \|x_1 - x\| + \|x - \bar{x}\| \leq (1 + \nu\kappa)r_{\bar{x}} + \kappa r_{\bar{y}} + r_{\bar{x}} = (2 + \nu\kappa)r_{\bar{x}} + \kappa r_{\bar{y}}. \tag{3.7}$$

Since $\nu\kappa < 1$, we have from the first condition in (3.1) that

$$(2 + \nu\kappa)r_{\bar{x}} + \kappa r_{\bar{y}} < \frac{2r_{\bar{x}} + \kappa r_{\bar{y}}}{1 - \nu\kappa} \leq r^*.$$

Thus, from (3.7), we have

$$\|x_1 - \bar{x}\| \leq r^*.$$

This implies that $x_1 \in \mathbb{B}_{r^*}(\bar{x})$. By using (3.7), we get

$$\begin{aligned}\|(y - f(x_1)) - \bar{y}\| &= \|y - \bar{y} + f(\bar{x}) - f(x_1)\| \leq \|y - \bar{y}\| + \|f(x_1) - f(\bar{x})\| \\ &\leq \|y - \bar{y}\| + \nu\|x_1 - \bar{x}\| \leq r_{\bar{y}} + \nu[(2 + \nu\kappa)r_{\bar{x}} + \kappa r_{\bar{y}}] \\ &= 2\nu r_{\bar{x}} + r_{\bar{y}} + \nu\kappa(\nu r_{\bar{x}} + r_{\bar{y}}). \end{aligned} \tag{3.8}$$

From the second condition in (3.1), we get $2\nu r_{\bar{x}} + r_{\bar{y}} \leq r^*(1 - \nu\kappa)$ and since $(1 - \nu\kappa)$ is positive, so $2\nu r_{\bar{x}} + r_{\bar{y}} \leq r^*$ implies that $\nu r_{\bar{x}} + r_{\bar{y}} \leq r^*$. Thus, we get from (3.8) that

$$\|(y - f(x_1)) - \bar{y}\| \leq r^*(1 - \nu\kappa) + \nu\kappa r^* = r^*.$$

This shows that $y - f(x_1) \in \mathbb{B}_{r^*}(\bar{y})$. Since F has locally closed graph, there exists $x_2 \in F^{-1}(y - f(x_1))$ and it is clear that (3.2) is true for $k = 1$. Also, since F is metrically regular and $x_0 = x$, we obtain

$$\begin{aligned}\|x_2 - x\| &\leq \text{dist}\Big(x, F^{-1}(y - f(x_1))\Big) \leq \kappa \ \text{dist}\Big(y - f(x_1), F(x)\Big) \\ &\leq \kappa \ \text{dist}\Big(y - f(x_1), y - f(x)\Big) \leq \nu\kappa\|x_1 - x\|. \end{aligned} \tag{3.9}$$

From (3.6) and (3.9), we get

$$\begin{aligned}\|x_2 - \bar{x}\| &\leq \|x_2 - x\| + \|x - \bar{x}\| \leq \nu\kappa\|x_1 - x\| + r_{\bar{x}} \leq \nu\kappa\Big[(1 + \nu\kappa)r_{\bar{x}} + \kappa r_{\bar{y}}\Big] + r_{\bar{x}} \\ &= \Big(1 + \nu\kappa + (\nu\kappa)^2\Big)r_{\bar{x}} + (\nu\kappa)\kappa r_{\bar{y}} = \frac{1}{1 - \nu\kappa}r_{\bar{x}} + (\nu\kappa)\kappa r_{\bar{y}}. \end{aligned} \tag{3.10}$$

Since $\frac{1}{1 - \nu\kappa} < \frac{2}{1 - \nu\kappa}$ and $\nu\kappa < \frac{1}{1 - \nu\kappa}$ for all values of $\nu\kappa$ such that $\nu\kappa < 1$, so by using the first condition in (3.1), we take the decision from (3.10) that

$$\begin{aligned}\|x_2 - \bar{x}\| &< \frac{2}{1 - \nu\kappa}r_{\bar{x}} + \frac{1}{1 - \nu\kappa}\kappa r_{\bar{y}} \\ &= \frac{2r_{\bar{x}} + \kappa r_{\bar{y}}}{1 - \nu\kappa} \leq r^*. \end{aligned}$$

This implies that $x_2 \in \mathbb{B}_{r^*}(\bar{x})$. Using the metric regularity condition on F, we obtain

$$\begin{aligned}\|x_2 - x_1\| &\leq \text{dist}\Big(x_1, F^{-1}(y - f(x_1))\Big) \leq \kappa \ \text{dist}\Big(y - f(x_1), F(x_1)\Big) \\ &\leq \kappa \ \text{dist}\Big(y - f(x_1), y - f(x)\Big) \leq \nu\kappa\|x_1 - x\|. \end{aligned}$$

This shows that (3.3) is true for $k = 1$. Thus, we have obtained two constructed points x_1, x_2 for which (3.2) and (3.3) are true for $k = 0, 1$. We assume that $x_1, x_2, ..., x_n$ are constructed such that (3.2) and (3.3) are true for $k = 0, 1, 2, .., n-1$. By induction hypothesis, we have to construct x_{n+1} such that (3.2) and (3.3) hold for $k = n$.

We will first show that $x_i \in \mathbb{B}_{r^*}(\bar{x})$ for all $i = 1, 2, ..., n$. By using (3.3), for such an i, we have

$$\|x_i - x\| \leq \sum_{j=0}^{i-1} \|x_{j+1} - x_j\| \leq \sum_{j=0}^{i-1} (\nu\kappa)^j \|x_1 - x\| \leq \frac{1}{1-\nu\kappa} \|x_1 - x\|. \tag{3.11}$$

Again, using (3.11), (3.6) and the first condition in (3.1), we obtain

$$\begin{aligned} \|x_i - \bar{x}\| &\leq \|x_i - x\| + \|x - \bar{x}\| \leq \frac{1}{1-\nu\kappa} \|x_1 - x\| + \|x - \bar{x}\| \\ &\leq \frac{1}{1-\nu\kappa} \Big[(1+\nu\kappa)r_{\bar{x}} + \kappa r_{\bar{y}}\Big] + r_{\bar{x}} = \frac{2r_{\bar{x}} + \kappa r_{\bar{y}}}{1-\nu\kappa} \leq r^*. \end{aligned} \tag{3.12}$$

This implies that $x_i \in \mathbb{B}_{r^*}(\bar{x})$ for all $i = 1, 2, ..., n$. Using (3.12) for $i = n$ and by the second condition in (3.1), we get

$$\begin{aligned} \|(y - f(x_n)) - \bar{y}\| &\leq \|y - \bar{y}\| + \|f(\bar{x}) - f(x_n)\| \leq \|y - \bar{y}\| + \nu\|x_n - \bar{x}\| \\ &\leq r_{\bar{y}} + \nu\Big(\frac{2r_{\bar{x}} + \kappa r_{\bar{y}}}{1-\nu\kappa}\Big) = \frac{2\nu r_{\bar{x}} + r_{\bar{y}}}{1-\nu\kappa} \leq r^*. \end{aligned}$$

This shows that $y - f(x_n) \in \mathbb{B}_{r^*}(\bar{y})$. Since F has locally closed graph, there exists $x_{n+1} \in F^{-1}(y - f(x_n))$ and it shows that (3.2) holds for $k = n$. Using the metric regularity condition on F, we obtain

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \operatorname{dist}\Big(x_n, F^{-1}(y - f(x_n))\Big) \leq \kappa \operatorname{dist}\Big(y - f(x_n), F(x_n)\Big) \\ &\leq \kappa \operatorname{dist}\Big(y - f(x_n), y - f(x_{n-1})\Big) \leq \kappa \|f(x_n) - f(x_{n-1})\| \\ &\leq \nu\kappa \|x_n - x_{n-1}\| \leq (\nu\kappa)^n \|x_1 - x\|. \end{aligned} \tag{3.13}$$

The induction steps are completed, and therefore (3.2) and (3.3) are satisfied for all k. By (3.13) with $x_0 = x$, we get

$$\|x_{n+1} - x\| \leq \sum_{i=0}^{n} \|x_{i+1} - x_i\| \leq \sum_{i=0}^{n} (\nu\kappa)^i \|x_1 - x\| \leq \frac{1}{1-\nu\kappa} \|x_1 - x\|. \tag{3.14}$$

By (3.14) and the relation $\frac{1}{1-\nu\kappa} \|x_1 - x\| + \|x - \bar{x}\| \leq r^*$ from (3.12), we obtain

$$\|x_{n+1} - \bar{x}\| \leq \|x_{n+1} - x\| + \|x - \bar{x}\| \leq \frac{1}{1-\nu\kappa} \|x_1 - x\| + \|x - \bar{x}\| \leq r^*.$$

This shows that $x_{n+1} \in \mathbb{B}_{r^*}(\bar{x})$. Since $\nu\kappa < 1$, we see from (3.13) that the sequence $\{x_k\}$ is a Cauchy sequence, and all its elements are in $\mathbb{B}_{r^*}(\bar{x})$. Hence, this sequence converges to some $\hat{x} \in \mathbb{B}_{r^*}(\bar{x})$, that is, $\hat{x} = \lim_{k\to\infty} x_k$. Then taking limit in (3.2) and the local closedness of gph$F$, satisfies $\hat{x} \in F^{-1}(y - f(\hat{x}))$, that is, $\hat{x} \in (f + F)^{-1}(y)$.

Moreover, by using (3.3) and (3.5), we obtain

$$\begin{aligned} \operatorname{dist}\Big(x, (f+F)^{-1}(y)\Big) &\leq \|\hat{x} - x\| = \lim_{k\to\infty} \|x_k - x\| \leq \lim_{k\to\infty} \sum_{i=0}^{k} \|x_{i+1} - x_i\| \\ &\leq \lim_{k\to\infty} \sum_{i=0}^{k} (\nu\kappa)^i \|x_1 - x\| \leq \frac{1}{1-\nu\kappa} \|x_1 - x\| \\ &\leq \frac{\kappa}{1-\nu\kappa} \operatorname{dist}\Big(y, (f+F)(x)\Big). \end{aligned}$$

Therefore the proof of the Lemma 3.1 is completed. □

Choose a sequence of scalars $\{\lambda_k\} \subseteq (0, \lambda)$. For each $x \in X$, define a mapping $H_{(\lambda_k, x)} : X \to Y$ by

$$H_{(\lambda_k,x)}(\cdot) = -\lambda_k(\cdot - x), \tag{3.15}$$

and a set valued mapping $\psi_{(\lambda_k,x)} : X \rightrightarrows 2^X$ by

$$\psi_{(\lambda_k,x)}(\cdot) = (f + F)^{-1}[H_{(\lambda_k,x)}(\cdot)]. \tag{3.16}$$

Here we present the statement and a proof of our vital result, which ensures the existence and the semi-local convergence of any sequence generated by the Gauss-type proximal point algorithm by using some sufficient conditions with initial point $\bar{x}$:

Theorem 3.1. *Suppose $\eta > 1$ and that $(f + F)$ is metrically regular at $(\bar{x}, \bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant $\dfrac{\kappa}{1-\nu\kappa}$ and $\operatorname{gph}(f + F) \cap (\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y}))$ is closed. Let $\delta > 0$ be such that*

(a) $\delta \leq \min\left\{\dfrac{r_{\bar{x}}}{2}, \dfrac{r_{\bar{y}}}{(3\eta + 1)\lambda}, 1\right\}$,

(b) $3\eta\kappa\lambda + \nu\kappa \leq 1$,

(c) $\|\bar{y}\| < \lambda\delta$.

Suppose that

$$\lim_{x\to\bar{x}} \operatorname{dist}\Big(\bar{y}, f(x) + F(x)\Big) = 0. \tag{3.17}$$

Then, with initial point $\bar{x}$, there exists some $\hat{\delta} > 0$ such that Algorithm 2 generates at least one sequence and any generated sequence $\{x_k\}$ converges to a solution $x^* \in \mathbb{B}_{\hat{\delta}}(\bar{x})$ of (1.1), that is, x^* satisfies that $0 \in f(x^*) + F(x^*)$.

Proof. It is sufficient to show that Algorithm 2 generates at least one sequence and any generated sequence $\{x_k\}$ satisfies

$$\|x_k - \bar{x}\| \leq 2\delta, \tag{3.18}$$

and

$$\|x_{k+1} - x_k\| \leq \left(\frac{1}{2}\right)^{k+1}\delta. \tag{3.19}$$

We will proceed by mathematical induction. For this aim, we define, for each $x \in X$,

$$r_{(\lambda,x)} := \frac{3\kappa}{2(1-\nu\kappa)}\Big(\|\bar{y}\| + \lambda\|x - \bar{x}\|\Big). \tag{3.20}$$

Since $\eta > 1$, by the conditions (b) and (c) we have, for each $x \in \mathbb{B}_{2\delta}(\bar{x})$,

$$r_{(\lambda,x)} \leq \frac{3\kappa}{2(1-\nu\kappa)}3\lambda\delta \leq \frac{3}{2\eta}\delta < \frac{3}{2}\delta < 2\delta. \tag{3.21}$$

Take $0 < \hat{\delta} \leq \delta$ such that

$$\operatorname{dist}\Big(0, f(x_0) + F(x_0)\Big) \leq \lambda\delta \quad \text{for each } x_0 \in \mathbb{B}_{\hat{\delta}}(\bar{x}) \tag{3.22}$$

(nothing that such $\hat{\delta}$ exists by (3.17) and assumption (c)). We see that (3.18) is obviously true for $k = 0$. In order to show (3.19) is valid for $k = 0$, it is sufficient to prove that the point x_1 exists, that is, $D(\lambda_0, x_0) \neq \emptyset$. To complete this, we have to prove that $D(\lambda_0, x_0) \neq \emptyset$ by applying Lemma 2.2 to the mapping $\psi_{(\lambda_0,x_0)}$ with $\eta_0 = \bar{x}$, $r := r_{(\lambda,x_0)}$ and $\alpha := \dfrac{1}{3}$. Below we show that assertions (2.1) and

(2.2) of Lemma 2.2 are satisfied with $\eta_0 = \bar{x}$, $r := r_{(\lambda,x_0)}$ and $\alpha := \frac{1}{3}$. Granting this, Lemma 2.2 is applicable to conclude that there exists a fixed point $\hat{x_1} \in \mathbb{B}_{r_{(\lambda,x_0)}}(\bar{x})$ such that $\hat{x_1} \in \psi_{(\lambda_0,x_0)}(\hat{x_1})$, which implies that $H_{(\lambda_0,x_0)}(\hat{x_1}) \in (f+F)(\hat{x_1})$, that is, $0 \in \lambda_0(\hat{x_1} - x_0) + (f+F)(\hat{x_1})$.

To proceed, note that $\bar{x} \in (f+F)^{-1}(\bar{y}) \cap \mathbb{B}_{r_{(\lambda,x_0)}}(\bar{x})$. By using the definition of excess e with the mapping $\psi_{(\lambda_0,x_0)}$ in (3.16) and using the relations $\mathbb{B}_{r_{(\lambda,x_0)}}(\bar{x}) \subseteq \mathbb{B}_{2\delta}(\bar{x}) \subseteq \mathbb{B}_{r_{\bar{x}}}(\bar{x})$, we have

$$\begin{aligned} \mathrm{dist}\Big(\bar{x}, \psi_{(\lambda_0,x_0)}(\bar{x})\Big) &\leq e\Big((f+F)^{-1}(\bar{y}) \cap \mathbb{B}_{r_{(\lambda,x_0)}}(\bar{x}), \psi_{(\lambda_0,x_0)}(\bar{x})\Big) \\ &\leq e\Big((f+F)^{-1}(\bar{y}) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), (f+F)^{-1}[H_{(\lambda_0,x_0)}(\bar{x})]\Big). \end{aligned} \tag{3.23}$$

Since $(3\eta+1)\lambda\delta \leq r_{\bar{y}}$ from the second relation in assumption (a), so $4\lambda\delta \leq r_{\bar{y}}$ (as $\eta > 1$). Also, using the relation $x_0 \in \mathbb{B}_{\hat{\delta}}(\bar{x}) \subseteq \mathbb{B}_{\delta}(\bar{x})$ and assumption (c), we obtain that,

$$\begin{aligned} \|H_{(\lambda_0,x_0)}(\bar{x}) - \bar{y}\| &= \|-\lambda_0(\bar{x} - x_0) - \bar{y}\| \leq \lambda_0\|x_0 - \bar{x}\| + \|\bar{y}\| \\ &\leq \lambda\|x_0 - \bar{x}\| + \|\bar{y}\| \\ &\leq 4\lambda\delta \leq r_{\bar{y}}. \end{aligned} \tag{3.24}$$

This shows that $H_{(\lambda_0,x_0)}(\bar{x}) \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$. Thus, by using (3.24), (3.20) and Lemma 2.1, we obtain from (3.23) that

$$\begin{aligned} \mathrm{dist}\Big(\bar{x}, \psi_{(\lambda_0,x_0)}(\bar{x})\Big) &\leq \frac{\kappa}{1-\nu\kappa}\|\bar{y} - H_{(\lambda_0,x_0)}(\bar{x})\| \\ &\leq \frac{\kappa}{1-\nu\kappa}\Big(\lambda\|x_0 - \bar{x}\| + \|\bar{y}\|\Big) = \Big(1 - \frac{1}{3}\Big)r_{(\lambda,x_0)} = (1-\alpha)r. \end{aligned}$$

It shows that assertion (2.1) of Lemma 2.2 hold. Now, we show that assertion (2.2) of Lemma 2.2 also hold. Let $x', x'' \in \mathbb{B}_{r_{(\lambda,x_0)}}(\bar{x})$. Thus, by the first relation $2\delta \leq r_{\bar{x}}$ from assumption (a) and $r_{(\lambda,x_0)} \leq 2\delta$ from (3.21), we have $x', x'' \in \mathbb{B}_{r_{(\lambda,x_0)}}(\bar{x}) \subseteq \mathbb{B}_{2\delta}(\bar{x}) \subseteq \mathbb{B}_{r_{\bar{x}}}(\bar{x})$. By the second relation in assumption (a), we get $4\lambda\delta \leq r_{\bar{y}}$ (as $\eta > 1$) and by using the assumption (c) and the relation $x_0 \in \mathbb{B}_{\hat{\delta}}(\bar{x}) \subseteq \mathbb{B}_{\delta}(\bar{x})$, we observe that

$$\begin{aligned} \|H_{(\lambda_0,x_0)}(x') - \bar{y}\| &= \|-\lambda_0(x' - x_0) - \bar{y}\| \leq \lambda\|x' - x_0\| + \|\bar{y}\| \\ &\leq \lambda\|x' - \bar{x}\| + \lambda\|\bar{x} - x_0\| + \|\bar{y}\| \leq 4\lambda\delta \leq r_{\bar{y}}. \end{aligned}$$

Hence $H_{(\lambda_0,x_0)}(x') \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$. Similarly, $H_{(\lambda_0,x_0)}(x'') \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$. So, by Lemma 2.1, we obtain that

$$\begin{aligned} e\Big(\psi_{(\lambda_0,x_0)}(x') \cap \mathbb{B}_{r_{(\lambda,x_0)}}(\bar{x}), \psi_{(\lambda_0,x_0)}(x'')\Big) &\leq e\Big(\psi_{(\lambda_0,x_0)}(x') \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), \psi_{(\lambda_0,x_0)}(x'')\Big) \\ &= e\Big((f+F)^{-1}[H_{(\lambda_0,x_0)}(x')] \\ &\cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), (f+F)^{-1}[H_{(\lambda_0,x_0)}(x'')]\Big) \\ &\leq \frac{\kappa}{1-\nu\kappa}\|H_{(\lambda_0,x_0)}(x') - H_{(\lambda_0,x_0)}(x'')\| \\ &= \frac{\kappa}{1-\nu\kappa}\lambda_0\|x' - x''\| \\ &\leq \frac{\lambda\kappa}{1-\nu\kappa}\|x' - x''\|. \end{aligned} \tag{3.25}$$

By assumption (b) and since $\eta > 1$, (3.25) becomes

$$e\Big(\psi_{(\lambda_0,x_0)}(x') \cap \mathbb{B}_{r_{(\lambda,x_0)}}(\bar{x}), \psi_{(\lambda_0,x_0)}(x'')\Big) \leq \frac{1}{3\eta}\|x' - x''\| < \frac{1}{3}\|x' - x''\| = \alpha\|x' - x''\|.$$

This implies that assertion (2.2) of Lemma 2.2 also hold. Thus, both the assertions of fixed point Lemma 2.2 hold, so we can deduce that there exists a fixed point $\hat{x_1} \in \mathbb{B}_{r_{(\lambda,x_0)}}(\bar{x})$ such that $\hat{x_1} \in \psi_{(\lambda_0,x_0)}(\hat{x_1})$. Therefore, $D(\lambda_0, x_0) \neq \emptyset$, and consequently, we can choose $d_0 \in D(\lambda_0, x_0)$ such that

$$\begin{aligned} \|d_0\| &\leq \eta \operatorname{dist}\Big(0, D(\lambda_0, x_0)\Big) \\ &\leq \eta r_{(\lambda,x_0)} \leq 2\eta\delta. \end{aligned} \tag{3.26}$$

By Algorithm 2, $x_1 := x_0 + d_0$ is defined. By the definition of $D(\lambda_0, x_0)$, we get

$$D(\lambda_0, x_0) := \Big\{d_o \in X : 0 \in \lambda_0 d_0 + f(x_0 + d_0) + F(x_0 + d_0)\Big\}$$

$$= \Big\{d_o \in X : x_0 + d_0 \in (f + F)^{-1}(-\lambda_0 d_0)\Big\}.$$

Thus, we have

$$\operatorname{dist}\Big(0, D(\lambda_0, x_0)\Big) = \operatorname{dist}\Big(x_0, (f + F)^{-1}(-\lambda_0 d_0)\Big). \tag{3.27}$$

By the choice of d_0 and the second relation $(3\eta + 1)\lambda\delta \leq r_{\bar{y}}$ in assumption (a) and (c), we obtain

$$\| - \lambda_0 d_0 - \bar{y}\| \leq \lambda\|d_0\| + \|\bar{y}\| \leq 2\lambda\eta\delta + \lambda\delta \leq r_{\bar{y}},$$

and so $-\lambda_0 d_0 \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$. Since $(f + F)$ is metrically regular at $(\bar{x}, \bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant $\dfrac{\kappa}{1 - \nu\kappa}$, we have from (3.26) and (3.27) that

$$\begin{aligned} \|d_0\| &\leq \eta \operatorname{dist}\Big(x_0, (f + F)^{-1}(-\lambda_0 d_0)\Big) \leq \frac{\eta\kappa}{1 - \nu\kappa}\operatorname{dist}\Big(- \lambda_0 d_0, f(x_0) + F(x_0)\Big) \\ &\leq \frac{\eta\kappa}{1 - \nu\kappa}\ \| - \lambda_0 d_0 - 0\| + \frac{\eta\kappa}{1 - \nu\kappa}\ \|0 - \Big(f(x_0) + F(x_0)\Big)\| \\ &\leq \frac{\eta\kappa\lambda}{1 - \nu\kappa}\|d_0\| + \frac{\eta\kappa}{1 - \nu\kappa}\ \operatorname{dist}\Big(0, f(x_0) + F(x_0)\Big). \end{aligned} \tag{3.28}$$

Using (3.22) in (3.28), we get

$$\|d_0\| \leq \frac{\eta\kappa\lambda}{1 - \nu\kappa}\|d_0\| + \frac{\eta\kappa}{1 - \nu\kappa}\lambda\delta. \tag{3.29}$$

Using assumption (b) in (3.29), we get

$$\|x_1 - x_0\| = \|d_0\| \leq \frac{\eta\kappa\lambda}{1 - \nu\kappa - \eta\kappa\lambda}\delta \leq \frac{1}{2}\delta.$$

This shows that (3.19) holds for $k = 0$. Assume that $x_1, \ldots, x_n$ are generated by Algorithm 2 and (3.18) and (3.19) are verified for $k = 0, 1, 2, \ldots, n - 1$. So, we obtain

$$\|x_n - \bar{x}\| \leq \sum_{i=0}^{n-1} \|x_{i+1} - x_i\| + \|x_0 - \bar{x}\| \leq \delta \sum_{i=0}^{n-1} \Big(\frac{1}{2}\Big)^{i+1} + \delta \leq 2\delta. \tag{3.30}$$

Thus, (3.18) is valid for $k = n$. We have to show that there exists a point x_{n+1} such that (3.19) is valid for $k = n$. In the similar way, as we did for the case of $k = 0$, we obtain by using Algorithm

2 that

$$\begin{aligned}
\|x_{n+1}-x_n\| = \|d_n\| &\le \eta \ \mathrm{dist}\Big(x_n, (f+F)^{-1}(-\lambda_n d_n)\Big)\\
&\le \frac{\eta\kappa}{1-\nu\kappa}\mathrm{dist}\Big(-\lambda_n d_n, f(x_n)+F(x_n)\Big)\\
&\le \frac{\eta\kappa}{1-\nu\kappa}\ \|-\lambda_n d_n - \Big(f(x_n)+F(x_n)\Big)\|\\
&\le \frac{\eta\kappa\lambda_n}{1-\nu\kappa}\ \|d_n\| + \frac{\eta\kappa}{1-\nu\kappa}\ \|f(x_n)+F(x_n)\|\\
&\le \frac{\eta\kappa\lambda}{1-\nu\kappa}\ \|d_n\| + \frac{\eta\kappa\lambda_{n-1}}{1-\nu\kappa}\ \|-\lambda_{n-1}(x_n-x_{n-1})\|\\
&\le \frac{\eta\kappa\lambda}{1-\nu\kappa}\|d_n\| + \frac{\eta\kappa\lambda}{1-\nu\kappa}\|x_n-x_{n-1}\|\\
&\le \frac{\eta\kappa\lambda}{1-\nu\kappa-\eta\kappa\lambda}\|x_n-x_{n-1}\| \qquad (3.31)\\
&\le \frac{1}{2}\cdot\Big(\frac{1}{2}\Big)^n\delta = \Big(\frac{1}{2}\Big)^{n+1}\delta.
\end{aligned}$$

Hence (3.19) is valid for $k = n$ and so (3.18) and (3.19) are valid for all k. This implies that $\{x_k\}$ is a Cauchy sequence and hence it is convergent, say, to x^*. So there exists $x^* \in \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ such that $x^* := \lim_{k\to\infty}(x_k)$. Now, the closedness of $\mathrm{gph}(f+F)\cap(\mathbb{B}_{r_{\bar{x}}}(\bar{x})\times\mathbb{B}_{r_{\bar{y}}}(\bar{y}))$ yields that $0 \in f(x^*)+F(x^*)$. Hence, the proof is completed. □

In the case where $\bar{x}$ is a solution of (1.1), that is, $\bar{y} = 0$, in Theorem 3.1, we have the following corollary, which gives the local convergence result of the G-PPA.

Corollary 3.1. *Suppose that $\eta > 1$, $\lambda > 0$, and let $\bar{x}$ be a solution of (1.1). Let $\mathrm{gph}(f+F)$ be locally closed at $(\bar{x}, 0)$ and let $(f+F)$ be metrically regular at $(\bar{x}, 0)$ with constant $\frac{\kappa}{1-\nu\kappa}$. Choose a sequence of scalars $\{\lambda_k\} \subseteq (0, \lambda)$. Suppose that*

$$\lim_{x\to\bar{x}} \mathrm{dist}\Big(0, f(x)+F(x)\Big) = 0. \qquad (3.32)$$

Then there exists $\hat{\delta} > 0$ such that any sequence $\{x_k\}$ generated by Algorithm 2 with initial point $x_0 \in \mathbb{B}_{\hat{\delta}}(\bar{x})$ converges to a solution x^* of (1.1), that is, x^* satisfies that $0 \in f(x^*)+F(x^*)$.

Proof. By hypothesis $(f+F)$ is metrically regular at $(\bar{x}, 0)$ which have locally closed graph at $(\bar{x}, 0)$ with constant $\frac{\kappa}{1-\nu\kappa}$. Then by definition there exist constants $r_{\bar{x}} > 0$ and $r_0 > 0$ such that $(f+F)$ is metrically regular at $(\bar{x}, 0)$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})\times\mathbb{B}_{r_0}(0)$ with constant $\frac{\kappa}{1-\nu\kappa}$, that is, the following inequality holds

$$\mathrm{dist}\Big(x, (f+F)^{-1}(y)\Big) \le \frac{\kappa}{1-\nu\kappa}\ \mathrm{dist}\Big(y, (f+F)(x)\Big) \quad \text{for all } x \in \mathbb{B}_{r_{\bar{x}}}(\bar{x}),\ y \in \mathbb{B}_{r_0}(0).$$

Let $\sup_k \lambda_k := \lambda \in (0, 1)$ be such that $3\eta\kappa\lambda + \nu\kappa \le 1$ and let $x_0 \in \mathbb{B}_{\hat{\delta}}(\bar{x})$. Since x_0 is very close to $\bar{x}$, then, for every y_0 near 0 such that $\mathrm{gph}(f+F)$ is locally closed at (x_0, y_0). Then (3.32) allow us to take $0 < \hat{\delta} \le \delta$ so that

$$\mathrm{dist}\Big(0, f(x_0)+F(x_0)\Big) \le \lambda\delta \quad \text{for each } x_0 \in \mathbb{B}_{\hat{\delta}}(\bar{x}).$$

Then, for each $0 < r \leq r_{\bar{x}}$ and $0 < \tilde{r} \leq r_0$, one has that

$$\text{dist}\Big(x, (f+F)^{-1}(y)\Big) \leq \frac{\kappa}{1-\nu\kappa}\,\text{dist}\Big(y, (f+F)(x)\Big) \quad \text{for all } x \in \mathbb{B}_r(\bar{x}),\ y \in \mathbb{B}_{\tilde{r}_{\bar{y}}}(\bar{y}),$$

that is, $(f+F)$ is metrically regular at $(\bar{x}, \bar{y})$ on $\mathbb{B}_r(\bar{x}) \times \mathbb{B}_{\tilde{r}_{\bar{y}}}(\bar{y})$ with constant $\dfrac{\kappa}{1-\nu\kappa}$. Choose $0 < r_1 < \dfrac{r_{\bar{x}}}{2}$ and $0 < r_2 < \dfrac{r_0}{2}$ be such that

$$\min\left\{\frac{r_1}{2}, \frac{r_2}{(3\eta+1)\lambda}\right\} > 0.$$

Thus, we can choose $0 < \delta \leq 1$ such that

$$\delta \leq \min\left\{\frac{r_1}{2}, \frac{r_2}{(3\eta+1)\lambda}\right\}.$$

Now, it is routine to check that all the assumptions in Theorem 3.1 hold. Thus, Theorem 3.1 is applicable to complete the proof of the corollary. □

4 Numerical Experiment

We introduce a numerical example in this section to verify the semi-local convergence result of the G-PPA generated by Algorithm 2.

Example 4.1. *Let* $X = Y = \mathbb{R}, x_0 = 0.5, \eta = 3, \lambda = 0.1, \nu = 0.4$ and $\kappa = 0.3$. *Define a differentiable function* f *on* $\mathbb{R}$ *by* $f(x) = 3x + 1$ *and a set-valued mapping* F *on* $\mathbb{R}$ *by* $F(x) = \{-7x+2, 4x-5\}$. *Then* $f+F$ *is a set-valued mapping on* $\mathbb{R}$ *defined by* $f(x)+F(x) = \{-4x+3, 7x-4\}$. *Then Algorithm 2 generates a sequence which converges to* $x^* = 0.75$.

Consider $f(x) + F(x) = -4x + 3$ and $\sup_k \lambda_k := \lambda = 0.1$. Then it is clear from the statement that $f + F$ is metrically regular at $(0.5, 1) \in \text{gph}(f + F)$. From (1.2), we obtain that

$$\begin{aligned} D(\lambda_k, x_k) &= \Big\{d_k \in \mathbb{R} : 0 \in \lambda_k(d_k) + f(x_k + d_k) + F(x_k + d_k)\Big\} \\ &= \Big\{d_k \in \mathbb{R} : d_k = \frac{10}{39}(3 - 4x_k)\Big\}. \end{aligned}$$

On the other hand, if $D(\lambda_k, x_k) \neq \emptyset$ we obtain that

$$0 \in \lambda_k(x_{k+1} - x_k) + f(x_{k+1}) + F(x_{k+1}) \Rightarrow x_{k+1} = \frac{30 - x_k)}{39}.$$

Thus from (3.31), we obtain that

$$\|d_k\| \leq \frac{\eta\kappa\lambda}{1 - \nu\kappa - \eta\kappa\lambda}\|d_{k-1}\|.$$

Since $\dfrac{\eta\kappa\lambda}{1-\nu\kappa-\eta\kappa\lambda} = \dfrac{9}{79} < 1$ for the given values of η, λ, κ and ν, thus we conclude that the sequence generated by Algorithm 2 converges linearly. The following table 1, obtained by using Matlab program, indicates that the solution of the generalized equation is 0.75 when $k = 5$.

Table 1. Finding a solution of generalized equation

x	f(x)+F(x)
0.5000	1.0000
0.7564	-0.0256
0.7498	0.0007
0.7500	-0.0000
0.7500	0.0000
0.7500	-0.0000

The following figure is the graphical representation of $f(x) + F(x)$

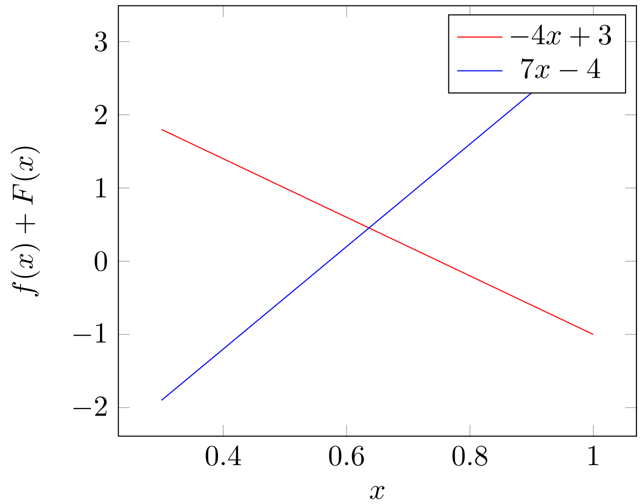

Fig. 1. The graph of $f(x) + F(x)$

5 Concluding Remarks

Under the assumptions that when f is a Fréchet differentiable function and F is metrically regular with $\eta > 1$, we have established semi-local and local convergence result for the G-PPA defined by Algorithm 2. Moreover, we have given a numerical example to verify the semi-local convergence result for Algorithm 2. If $\eta = 1$ and $D(\lambda_k, x_k)$ is singleton, Algorithm 2 is identical with the Algorithm 1 introduced by Dontchev and Rockafellar [10, Chapter 6]. If F is the normal cone mapping, $\lambda_k u = g_k(u)$ a sequence of Lipschitz continuous functions and $Y = X^*$ a dual Banach space of X, the results established in the present paper coincide with the results obtained in [3]. This result extends and improves the result obtained in [3, 10].

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Competing Interests

Authors have declared that no competing interests exist.

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