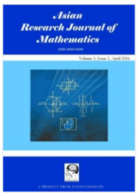

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# Influence of Rotatory Inertial Correction Factor on the Vibration of Elastically Supported Non-Uniform Rayleigh Beam on Variable Foundation

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***Authors' contributions***

*This work was carried out in collaboration between both authors. Author AAS designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript and managed literature searches. Author ATO managed the analyses of the study and literature searches. Both authors read and approved the final manuscript.*

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## Abstract

The influence of rotatory inertial correction factor on the vibration of elastically supported non-uniform Rayleigh beam under moving distributed masses and resting on variable bi-parametric elastic foundation is investigated. The governing equation is a fourth order partial differential equation with variable and singular co-efficients. In order to solve this equation, the method of Galerkin is used to reduce the governing differential equation to a sequence of coupled second order ordinary differential equation which is then simplified with modified asymptotic method of Struble. The simplified equation is solved using the integral transformation technique. The analysis of the closed form solution shows that resonance is attained earlier in moving mass system than in the moving force system. The results in plotted graphs show that as the rotatory inertia, foundation modulus and shear modulus increase, the deflection of the elastically supported non-uniform Rayleigh beam decreases in each case. The transverse deflections of the

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elastically supported beam on variable bi parametric elastic foundation are higher under the action of moving masses than those when only the force effects of the moving load are considered. This implies that resonance is reached faster in moving mass problem than in moving force problem.

*Keywords:* *Bi-parametric foundation; shear deformation, resonance; critical speed; natural frequency; modified frequency.*

# 1 Introduction

In recent years, while heavy and high speed vehicles are increasing in number due to increase in domestic and foreign trade, the development in design, and construction technology in civil engineering enable the construction of more light and slender structures, which causes structures to be more vulnerable to dynamic loads, especially moving loads. Large deflections and vibrations of beam-like structures especially bridges induced by the heavy and high speed vehicles may significantly increase the internal stresses and the safety and serviceability of the bridges.

Several authors [1-5] in engineering and applied mathematics have extensively studied the vibration of beams. The study of vibration of beams and that due to applied forces (moving or static) have not been neglected. The problem of moving load was first tackled for the case in which the beam mass was considered small against the mass of a single constant load. Stokes [6] approached the problem under similar assumptions. The other extreme case, i.e. that of load mass small against the beam mass, was originally examined for a simply supported beam and a constant concentrated force by Krylov [7] using the method of expansion of the eigenfunctions, and by Timoshenko [8]. Lowan [9] and Bondar [10] solved it with the aid of Green's functions and integral equations respectively.

The problem involving both the load mass and the beam mass, considerably more complicated than the preceding special cases, was not solved until much later.Jeffcott [11] whose iterative method becomes divergent in some cases.

In most cases, the classes of non-classical boundary value problems are in general resistant to the classical methods of solving dynamical problems. Obviously, it becomes more complex and cumbersome when the dynamical problem involves moving loads with or without consideration of the inertial effects of the moving loads is taken into consideration. Mindin and Goodman [12] considered a procedure for extending the method of separation of variables to the solution of Bernoulli-Euler beam vibration problems with time-dependent boundary conditions. Oni and Ogunyebi [13] considered the dynamic analysis of a pre-stressed elastic beam with general boundary conditions under the action of uniformly distributed masses. Biot [14] studied the bending of an infinite beam resting on elastic foundation. Omolofe [15] investigated the deflection of beams resting on two parameter elastic foundation. Oni and Awodola [16] investigated behavior of moving concentrated masses of simply supported rectangular plates on variable Winkler elastic foundation. Awodola abnd Oni [17] investigated the dynamic response to moving masses of rectangular plates with general boundary conditions resting on one variable foundation(i.e a Winkler foundation). Teodoru,Musat and Vrabie [18] carried out study on bending behavior of beams resting on two parameter elastic foundation by using a finite element method. In the same vein, Oni and Ogunyebi [19] studied the dynamic analysis of a pre-stressed elastic beam with general boundary conditions resting on a bi-parametric foundation. In order to improve on [18], Teodoru [20] and [21] used the finite difference approach and the simplified continuous approach to analysis the behavior of beams resting on two parameter foundations. Also, Agboola and Gbadeyan [22] considered the dynamic behavior of a double Rayleigh beam under the influence of uniform partially distributed load.

Though these works are impressive, the non-uniform beams were only subjected to a concentrated moving mass under Winkler foundation, thus limiting the scope of applicability of the study since it is well known that the Winkler foundation predicts discontinuities in the deflections of the surface of the foundation at the end of a finite beam and in reality, the surface displacements continue beyond the load (force) region. Some researchers even considered beams resting on variable elastic foundation but never used the elastically supported boundary conditions used in this research work.

In all the aforementioned, works are based on structures with classical conditions, where the non-classical boundary conditions are considered, the loads are considered to be concentrated loads and the foundation taken to be constant.This work however, considers the influence of rotatory inertial correction factor on the vibration of elastically supported non-uniform Rayleigh beam under moving distributed load and resting on bi-parametric variable foundation.

# 2 Governing Equation

Considering the influence of rotatory inertial correction factor on the vibration of elastically supported Rayleigh beam on variable bi-parametric elastic foundation; the governing equation of motion is given by the fourth order partial differential equation Fryba [1].

$$\frac{\partial^2}{\partial x^2}[EI\frac{\partial^2 V(x,t)}{\partial x^2}] - N_0\frac{\partial^2 V(x,t)}{\partial x^2} + \mu(x)\frac{\partial^2 V(x,t)}{\partial t^2} - \mu(x)R^0\frac{\partial^4 V(x,t)}{\partial x^2 \partial t^2} + G_f(x,t) = P(x,t). \quad (2.1)$$

where $x$ is the spatial co-ordinate, $t$ is the time co-ordinate,$V(x,t)$ is the transverse displacement, $EI(x)$ is the variable flexural rigidity of the structure, $\mu(x)$ is the variable mass per unit length of the non-uniform beam, $N_0$ is the constant axial force, $R^0$ is the rotatory inertial correction factor, $G_f(x,t)$ is the variable foundation reaction,$P(x,t)$ is the moving distributed load.
The relationship between the foundation reaction and lateral deflection $V(x,t)$ is

$$G_f(x,t) = S(x,t)V(x,t) - \frac{\partial}{\partial x}[K(x)\frac{\partial V(x,t)}{\partial x}] \quad (2.2)$$

$$\mu(x) = \mu_0(1+\sin\frac{\pi x}{L}), \quad I(x) = I_0(1+\sin\frac{\pi x}{L})^3 \quad (2.3)$$

where $S(x)$ and $K(x)$ are two variable parameters of the elastic foundation. That is, $S(x)$ is the variable foundation stiffness and $K(x)$ is the variable shear modulus.

where

$$S(x) = S_0(4x - 3x^2 + x^3) \quad (2.4)$$

$$K(x) = K_0(12 - 13x + 6x^2 + x^3) \quad (2.5)$$

substituting equations (2.3), (2.4) and (2.5) into equation (2.1), one obtains

$$\frac{\partial^2}{\partial x^2}\left[EI_0(1+\sin\frac{\pi x}{L})^3\frac{\partial^2}{\partial x^2}V(x,t)\right] - N_0\frac{\partial^2}{\partial x^2}V(x,t) + \mu_0\left(1+\sin\frac{\pi x}{L}\right)\frac{\partial^2}{\partial t^2}V(x,t) + \mu_0\left(1+\sin\frac{\pi x}{L}\right)$$
$$R_0\frac{\partial^4}{\partial x^2\partial t^2}V(x,t) + S_0\left(4x-3x^2+x^3\right)V(x,t) - \frac{\partial}{\partial x}\left[K_0(12-13x+6x^2+x^3)\frac{\partial}{\partial x}V(x,t)\right]$$
$$= \sum_{i=1}^{N} MgH(x-ct)\left[1 - \frac{1}{g}\frac{\partial^2}{\partial t^2}V(x,t)\right] \quad (2.6)$$

Equation (2.6) can also be re-written as

$$EI_0\left[\frac{\partial^2}{\partial x^2}\left(10+15\sin\frac{\pi x}{L}-6\cos\frac{2\pi x}{L}-\sin\frac{3\pi x}{L}\right)\frac{\partial^2}{\partial x^2}V(x,t)\right]-N_0\frac{\partial^2}{\partial x^2}V(x,t)+\mu_0\left(1+\sin\frac{\pi x}{L}\right)$$
$$\frac{\partial^2}{\partial t^2}V(x,t)+\mu_0\left(1+\sin\frac{\pi x}{L}\right)R_0\frac{\partial^4}{\partial x^2\partial t^2}V(x,t)+s_0\left(4x-3x^2+x^3\right)V(x,t)-\frac{\partial}{\partial x}\left[K_0(12-13x+6x^2\right.$$
$$\left.+x^3)\frac{\partial}{\partial x}V(x,t)\right]+MH(x-ct)\left[\frac{\partial^2}{\partial t^2}+2c\frac{\partial^2}{\partial x\partial t}+c^2\frac{\partial^2}{\partial x^2}\right]V(x,t)=MHg(x-ct) \tag{2.7}$$

The boundary condition of the structure under consideration is first taken to be arbitrary and the inertial condition without any loss of generality is taken as

$$V(x,0)=0=\frac{\partial}{\partial x}V(x,0) \tag{2.8}$$

# 3 Analytical Approximate Solution

An exact closed form solution of the above fourth order partial differential equation (2.1) does not exit. Therefore,an approximate solution is sought. The Galerkin's method is employed, this technique requires the solution of equation (2.1) takes the form

$$V_m(x,t)=\sum_{m=1}^{N}W_m(t)U_m(x) \tag{3.1}$$

where

$$U_m(x)=\sin\frac{\lambda_m x}{L}+A_m\cos\frac{\lambda_m x}{L}+B_m\sinh\frac{\lambda_m x}{L}+C_m\cosh\frac{\lambda_m x}{L} \tag{3.2}$$

is the beam function chosen so that the concerned boundary conditions are satisfied.
Substituting equation (3.1) into equation (2.7), one obtains

$$\sum_{m=1}^{N}\left[U_m(x)U_k(x)+\sin\tfrac{\pi x}{L}U_m(x)U_k(x)-R_0\left(U''_m(x)U_k(x)+\sin\tfrac{\pi x}{L}U''_m(x)U_k(x)\right)\ddot{W}_m(t)+\left(\tfrac{EI_0}{4\mu_0}\right.\right.$$
$$\left[10U_m^{(iv)}(x)U_k(x)+\tfrac{15\pi x}{L}U_m^{(iv)}(x)U_k(x)+\tfrac{30\pi x}{L}\cos\tfrac{\pi x}{L}U'''_m(x)U_k(x)-\tfrac{15\pi^2}{L^2}\sin\tfrac{\pi x}{L}U''_m(x)U_k(x)-\right.$$
$$\left.6\cos\tfrac{\pi x}{L}U''_m(x)U_k(x)-\sin\tfrac{3\pi x}{L}U_m^{(iv)}(x)U_k(x)+\tfrac{9\pi^2}{L^2}\sin\tfrac{3\pi x}{L}U'''_m(x)U_k(x)\right]-\tfrac{N_0}{\mu_0}U''_m(x)U_k(x)+$$
$$\tfrac{S_0}{\mu_0}\left(4xU_m(x)U_k(x)-3x^2U_m(x)U_k(x)+x^3U_m(x)U_k(x)\right)-\tfrac{K_0}{\mu_0}\left(-13U'_m(x)U_k(x)+12xU'_m(x)U_k(x)\right.$$
$$\left.-3x^2U'_m(x)U_k(x)\right)-\tfrac{K_0}{\mu_0}\left(12U''_m(x)U_k(x)-13xU''_m(x)U_k(x)+\right.$$
$$\left.\left.6x^2U''_m(x)U_k(x)-x^3U''_m(x)U_k(x)\right)\right)W_m(t)+\tfrac{M}{\mu_0}H(x-ct)\left[U_m(x)U_k(x)\ddot{W}_m(t)+2cU'_m(x)U_k(x)\right.$$
$$\left.(x)\dot{W}_m(t)+c^2U''_m(x)U_k(x)W_m(t)\right]$$
$$=0 \tag{3.3}$$

In order to get $W_m(t)$, it is required that the expression on the left hand side equation (3.3) is orthogonal to the function $U_k(x)$, where k is the dummy index. Therefore, one obtains

$$\int_0^L \sum_{m=1}^{N} \Bigg( \Bigg[ U_m(x)U_k(x) + \sin\frac{\pi x}{L}U_m(x)U_k(x) - R_0\Big(U_m''(x)U_k(x) + \sin\frac{\pi x}{L}U_m''(x)U_k(x)\Big)\Bigg]\ddot{W}_m(t) +$$
$$\Bigg(\Bigg(\frac{EI_0}{4\mu_0}[10U_m^{(iv)}(x)U_k(x) + \frac{15\pi x}{L}U_m^{(iv)}(x)U_k(x) + \frac{30\pi x}{L}\cos\frac{\pi x}{L}U_m'''(x)U_k(x) - \frac{15\pi^2}{L^2}\sin\frac{\pi x}{L}U_m''(x)U_k(x)$$
$$-6\cos\frac{\pi x}{L}U_m''(x)U_k(x) - \sin\frac{\pi x}{L}U_m^{(iv)}(x)U_k(x) + \frac{9\pi^2}{L^2}\sin\frac{3\pi x}{L}U_m'''(x)U_k(x)] - \frac{N_0}{\mu_0}U_m''(x)U_k(x) + \frac{S_0}{\mu_0}$$
$$\Big(4xU_m(x)U_k(x) - 3x^2U_m(x)U_k(x) + x^3U_m(x)U_k(x)\Big) - \frac{K_0}{\mu_0}\Big( - 13U_m'(x)U_k(x) + 12xU_m'(x)U_k(x)$$
$$-3x^2U_m'(x)U_k(x)\Big) - \frac{K_0}{\mu_0}\Big(12U_m''(x)U_k(x) - 13xU_m''(x)U_k(x) + 6x^2U_m''(x)U_k(x) - x^3U_m''(x)U_k(x)\Big)$$
$$\Bigg)W_m(t) + \frac{M}{\mu_0}H(x-ct)\Bigg[U_m(x)U_k(x)\ddot{W}_m(t) + 2cU_m'(x)U_k(x)\dot{W}_m(t) + c^2U_m''(x)U_k(x)W_m(t)\Bigg]\Bigg)dx$$
$$= 0 \tag{3.4}$$

equation (3.4) can be re-written as

$$\sum_{m=1}^{N}\Bigg(\Bigg[B_0(m,k) + B_1(m,k) - R^0(B_2(m,k) + B_3(m,k))\Bigg]\ddot{W}_m(t) + Q_A\Bigg[10B_4(m,k) + 15B_5(m,k)$$
$$+ \frac{30\pi}{L}B_6(m,k) - \frac{15\pi^2}{L^2}B_7(m,k) - 6B_8(m,k) + \frac{24\pi}{L}B_9(m,k) + \frac{24\pi^2}{L^2}B_{10}(m,k) - \frac{6\pi}{L}B_{11}(m,k) -$$
$$B_{12}(m,k) + \frac{9\pi^2}{L^2}B_{13}(m,k)\Bigg] + Q_BB_{14}(m,k) + \frac{S_0}{\mu_0}\Bigg(4B_{15}(m,k) - 3B_{16}(m,k) + B_{17}(m,k)\Bigg) - \frac{K_0}{\mu_0}$$
$$\Bigg( - 13B_{18}(m,k) + 12B_{19}(m,k) - 3B_{20}(m,k)\Bigg) - \frac{K_0}{\mu_0}\Bigg(12B_{21}(m,k) - 13B_{22}(m,k) + 6B_{23}(m,k) -$$
$$3B_{24}(m,k)\Bigg)\Bigg]W_m(t) + \frac{M}{\mu_0}\Bigg[B_{25}(m,k)\ddot{W}_m(t) + 2cB_{26}(m,k)\dot{W}_m(t) + c^2B_{27}(m,k)W_m(t)\Bigg] -$$
$$\frac{Mg}{\mu_0}B_{28}(m,k)\Bigg) = 0 \tag{3.5}$$

where

$$Q_A = \frac{EI_0}{\mu_0}, B_0(m,k) = \int_0^L U_m(x)U_k(x)dx, B_1(m,k) = \int_0^L \sin\frac{\pi x}{L}U_m(x)U_k(x)dx \tag{3.6}$$
$$B_2(m,k) = \int_0^L U_m''(x)U_k(x)dx, B_3(m,k) = \int_0^L \sin\frac{\pi x}{L}U_m''(x)U_k(x)dx \tag{3.7}$$
$$B_4(m,k) = \int_0^L U_m^{(iv)}(x)U_k(x)dx, B_5(m,k) = \int_0^L \sin\frac{\pi x}{L}U_m^{(iv)}(x)U_k(x)dx \tag{3.8}$$
$$B_6(m,k) = \int_0^L \cos\frac{\pi x}{L}U_m'''(x)U_k(x)dx, B_7(m,k) = \int_0^L \sin\frac{\pi x}{L}U_m''(x)U_k(x)dx \tag{3.9}$$
$$B_8(m,k) = \int_0^L \cos\frac{\pi x}{L}U_m^{('')}(x)U_k(x)dx, B_9(m,k) = \int_0^L \sin\frac{\pi x}{L}U_m'''(x)U_k(x)dx \tag{3.10}$$

$$B_{10}(m,k) = \int_0^L \cos\frac{2\pi x}{L} U_m^{'''}(x)U_k(x)dx, B_{11}(m,k) = \int_0^L \cos\frac{3\pi x}{L} U_m^{'''}(x)U_k(x)dx \qquad (3.11)$$

$$B_{12}(m,k) = \int_0^L \sin\frac{3\pi x}{L} U_m^{'''}(x)U_k(x)dx, B_{13}(m,k) = \int_0^L \sin\frac{3\pi x}{L} U_m^{''}(x)U_k(x)dx \qquad (3.12)$$

$$B_{14}(m,k) = \int_0^L U_m^{''}(x)U_k(x)dx, B_{15}(m,k) = \int_0^L xU_m(x)U_k(x)dx \qquad (3.13)$$

$$B_{16}(m,k) = \int_0^L x^2U_m(x)U_k(x)dx, B_{17}(m,k) = \int_0^L x^3U_m(x)U_k(x)dx \qquad (3.14)$$

$$B_{18}(m,k) = \int_0^L U_m^{'}(x)U_k(x)dx, B_{19}(m,k) = \int_0^L xU_m^{'}(x)U_k(x)dx \qquad (3.15)$$

$$B_{20}(m,k) = \int_0^L x^2U_m^{'}(x)U_k(x)dx, B_{21}(m,k) = \int_0^L U_m^{''}(x)U_k(x)dx \qquad (3.16)$$

$$B_{22}(m,k) = \int_0^L xU_m^{''}(x)U_k(x)dx, B_{23}(m,k) = \int_0^L x^2U_m^{''}(x)U_k(x)dx \qquad (3.17)$$

$$B_{24}(m,k) = \int_0^L x^3U_m^{''}(x)U_k(x)dx, B_{25}(m,k) = \int_0^L H(x-ct)U_m(x)U_k(x)dx \qquad (3.18)$$

$$B_{26}(m,k) = \int_0^L H(x-ct)U_m^{'}(x)U_k(x)dx, B_{27}(m,k) = \int_0^L H(x-ct)U_m^{''}(x)U_k(x)dx \qquad (3.19)$$

$$B_{28}(m,k) = \int_0^L H(x-ct)U_k(x)dx \qquad (3.20)$$

In order to evaluate the integrals in equations (3.19) and (3.20), one makes use of the Fourier series representation for the Heaviside function in the form

$$H(x-ct) = \frac{1}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n+1)\pi(x-ct)}{2n+1}, 0 < x < 1 \qquad (3.21)$$

substituting equation (3.21) into equation (3.5), after some simplifications and re-arrangements, one obtains

$$\sum_{m=1}^{N}\Bigg[\Bigg(B_0(m,k)+B_1(m,k)-R^0\Bigg(B_2(m,k)+B_3(m,k)\Bigg)\Bigg)\ddot{W}_m(t)+Q_A\Bigg[10B_4(m,k)+15B_5(m,k)+ \frac{30\pi}{L}B_6(m,k)-\frac{15\pi^2}{L^2}B_7(m,k)-6B_8(m,k)+\frac{24\pi}{L}B_9(m,k)+\frac{24\pi^2}{L^2}B_{10}(m,k)-\frac{6\pi}{L}B_{11}(m,k)- B_{12}(m,k)+\frac{9\pi^2}{L^2}B_{13}(m,k)\Bigg]+Q_B\Bigg[B_{14}(m,k)+\frac{S_0}{\mu_0}\Bigg(4B_{15}(m,k)-3B_{16}(m,k)+B_{17}(m,k)\Bigg)-\frac{K_0}{\mu_0} \Bigg(-13B_{18}(m,k)+12B_{19}(m,k)-3B_{20}(m,k)\Bigg)-\frac{K_0}{\mu_0}\Bigg(12B_{21}(m,k)-13B_{22}(m,k)+6B_{23}(m,k)- 3B_{24}(m,k)\Bigg)\Bigg]W_m(t)+\frac{M}{\mu_0}\Bigg(B_{25}(m,k)\ddot{W}_m(t)+2cB_{26}(m,k)\dot{W}_(t)+c^2B_{27}(m,k)W_m(t)\Bigg)\Bigg]=\frac{MgL}{\mu_0\lambda_k} \Bigg[-\cos\lambda_k+A_k\sin\lambda_k+B_k\cosh\lambda_k+C_k\sinh\lambda_k+\cos\frac{\lambda_k ct}{L}-A_k\sin\frac{\lambda_k ct}{L}-B_k\cosh\frac{\lambda_k ct}{L} -C_k\sinh\frac{\lambda_k ct}{L}\Bigg] \qquad (3.22)$$

Therefore, equation (3.22) becomes

$$\omega_0(m,k)\ddot{W}_m(t) + \omega_1(m,k)W_m(t) + \gamma\Bigg[\Bigg(\tfrac{D_1(m,k)}{4} + \tfrac{1}{\pi}\textstyle\sum_{n=1}^{\infty} \tfrac{\cos\ (2n+1)\pi ct}{(2n+1)} D_2(n,m,k)$$

$$-\tfrac{1}{\pi}\textstyle\sum_{n=1}^{\infty} \tfrac{\sin(2n+1)\pi ct}{(2n+1)} D_3(n,m,k)\Bigg)\ddot{W}_m(t) + 2c\Bigg(\tfrac{D_4(m,k)}{4} + \tfrac{1}{\pi}\textstyle\sum_{n=1}^{\infty} \tfrac{\cos(2n+1)\pi ct}{(2n+1)} D_5(n,m,k) -$$

$$\tfrac{1}{\pi}\textstyle\sum_{n=1}^{\infty} \tfrac{\sin(2n+1)\pi ct}{(2n+1)} D_6(n,m,k)\Bigg)\dot{W}_m(t) + c^2\Bigg(\tfrac{D_7(m,k)}{4} + \tfrac{1}{\pi}\textstyle\sum_{n=1}^{\infty} \tfrac{\cos(2n+1)\pi ct}{(2n+1)} D_8(n,m,k) -$$

$$\tfrac{1}{\pi}\textstyle\sum_{n=1}^{\infty} \tfrac{\sin(2n+1)\pi ct}{(2n+1)} D_9(n,m,k)\Bigg)W_m(t)\Bigg] = \tfrac{PL}{\mu_0\lambda_k}\Bigg[-\cos\lambda_k$$

$$+A_k\sin\lambda_k + B_k\cosh\lambda_k + C_k\sinh\lambda_k + \cos\tfrac{\lambda_k ct}{L} - A_k\sin\tfrac{\lambda_k ct}{L} - B_k\cosh\tfrac{\lambda_k ct}{L} - C_k\sinh\tfrac{\lambda_k ct}{L}\Bigg] \tag{3.23}$$

where

$$\omega_0(m,k) = \Bigg[B_0(m,k) + B_1(m,k) - R^0\Bigg(B_2(m,k) + B_3(m,k)\Bigg)\Bigg] \tag{3.24}$$

$$\omega_1(m,k) = Q_A\Bigg[10B_4(m,k) + 15B_5(m,k) + \frac{30\pi}{L}B_6(m,k) - \frac{15\pi^2}{L^2}B_7(m,k) - 6B_8(m,k) +$$

$$\frac{24\pi}{L}B_9(m,k) + \frac{24\pi^2}{L^2}B_{10}(m,k) - \frac{6\pi}{L}B_{11}(m,k) - B_{12}(m,k) + \frac{9\pi^2}{L^2}B_{13}(m,k)\Bigg] + Q_B\Bigg[B_{14}(m,k) +$$

$$\frac{S_0}{\mu_0}[4B_{15}(m,k) - 3B_{16}(m,k) + B_{17}(m,k)\Bigg] - \frac{K_0}{\mu_0}\Bigg(-13B_{18}(m,k) + 12B_{19}(m,k) - 3B_{20}(m,k)\Bigg) -$$

$$\frac{K_0}{\mu_0}\Bigg(12B_{21}(m,k) - 13B_{22}(m,k) + 6B_{23}(m,k) - 3B_{24}(m,k)\Bigg)\Bigg] \tag{3.25}$$

$$\gamma = \frac{M}{\mu_0 L}, P = Mg \tag{3.26}$$

equation (3.24) is re-written as

$$\ddot{W}_m(t) + \frac{\omega_1(m,k)}{\omega_0(m,k)}W_m(t) + \frac{\gamma}{\omega_0(m,k)}\Bigg[\Bigg(\frac{D_1(m,k)}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos\ (2n+1)\pi ct}{(2n+1)}D_2(n,m,k) -$$

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin\ (2n+1)\pi ct}{(2n+1)}D_3(n,m,k)\Bigg)\ddot{W}_m(t) + 2c\Bigg(\frac{D_4(m,k)}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos\ (2n+1)\pi ct}{(2n+1)}D_5(n,m,k) -$$

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin\ (2n+1)\pi ct}{(2n+1)}D_6(n,m,k)\Bigg)\dot{W}_m(t) + c^2\Bigg(\frac{D_7(m,k)}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos\ (2n+1)\pi ct}{(2n+1)}D_8(n,m,k) -$$

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin\ (2n+1)\pi ct}{(2n+1)}D_9(n,m,k)\Bigg)W_m(t)\Bigg] = \frac{PL}{\mu_0\omega_0(m,k)\lambda_k}\Bigg[-\cos\lambda_k + A_k\sin\lambda_k +$$

$$B_k\cosh\lambda_k + C_k\sinh\lambda_k + \cos\frac{\lambda_k ct}{L} - A_k\sin\frac{\lambda_k ct}{L} - B_k\cosh\frac{\lambda_k ct}{L} - C_k\sinh\frac{\lambda_k ct}{L}\Bigg] \tag{3.27}$$

equation (3.26) is the transformed equation governing the problem of the vibration of the non-uniform Rayleigh beam under moving distributed masses and resting on variable bi-parametric elastic foundation. This coupled non-homogeneous second order ordinary differential equation is assumed to have arbitrary boundary conditions.

**Case I: Moving Force Problem**

In moving force problem, only the load being transferred to the structure. In this case, the inertia effect is negligible. Setting $\gamma = 0$ in the transformed equation (3.27),one obtains

$$\ddot{W}_m(t) + \frac{\omega_1(m,k)}{\omega_0(m,k)} W_m(t) = \frac{PL}{\mu_0\omega_0(m,k)\lambda_k}\Bigg[ -\cos\lambda_k + A_k \sin\lambda_k + B_k\cosh\lambda_k + C_k\sinh\lambda_k + \cos\frac{\lambda_k ct}{L} - A_k\sin\frac{\lambda_k ct}{L} - B_k\cosh\frac{\lambda_k ct}{L} - C_k\sinh\frac{\lambda_k ct}{L}\Bigg] \tag{3.28}$$

equation (3.28) can be re-written as,

$$\ddot{W}_m(t) + \theta_m^2 W_m(t) = \frac{PL}{\mu_0\omega_0(m,k)\lambda_k}\Bigg[\beta_k + \cos\frac{\lambda_k ct}{L} - A_k\sin\frac{\lambda_k ct}{L} - B_k\cosh\frac{\lambda_k ct}{L} - C_k\sinh\frac{\lambda_k ct}{L}\Bigg] \tag{3.29}$$

where

$$\theta_m^2 = \frac{\omega_1(m,k)}{\omega_0(m,k)}, \beta_k = -\cos\lambda_k + A_k\sin\lambda_k + B_k\cosh\lambda_k + C_k\sinh\lambda_k \tag{3.30}$$

equation (3.29) is an approximate model, which assumes the inertia effect of the moving mass as negligible.

Further re-arrangement of equation (3.29) yields

$$\ddot{W}_m(t) + \theta_m^2 W_m(t) = \frac{PL}{\mu_0\omega_0(m,k)\lambda_k}[\beta_k + \cos\alpha_k - A_k\sin\alpha_k - B_k\cosh\alpha_k - C_k\sinh\alpha_k] \tag{3.31}$$

where

$$\alpha_k = \frac{\lambda_k ct}{L} \tag{3.32}$$

solving equation (3.31) using Laplace transformation and convolution theory and taking into account equation (2.4), one obtains

$$V_m(x,t) = \frac{PL}{\mu_0\omega_0(m,k)\lambda_k} \times \frac{1}{\theta_m^2(\theta_m^4 - \alpha_k^4)}\Bigg[\beta_k(1-\cos\theta_m t)(\theta_m^4 - \alpha_k^4) + \theta_m^2(\theta_m^2 + \alpha_k^2)(\cos\alpha_k t - \cos\theta_m t) - A_k\theta_m(\theta_m^2 + \alpha_k^2)(\theta_m\sin\alpha_k t - \alpha_k\sin\theta_m t) - B_k\theta_m^2(\theta_m^2 - \alpha_k^2)(\cosh\alpha_k t - \cos\theta_m t) - C_k\theta_m(\theta_m^2 + \alpha_k^2)(\theta_m\sinh\alpha_k t - \alpha_k\sin\theta_m t)\Bigg] \times \left(\sin\frac{\lambda_m x}{L} + A_m\cos\frac{\lambda_m x}{L} + B_m\sinh\frac{\lambda_m x}{L} + C_m\cosh\frac{\lambda_m x}{L}\right) \tag{3.33}$$

equation (3.33) represents the transverse deflection of the non-uniform Rayleigh beam under moving distributed force and resting on variable Pasternak elastic foundation.

**Case II: Moving Mass Problem**

In moving mass problem, the moving load is assumed rigid, and the weight and as well as inertia forces are transferred to the moving load. That is, the inertia effect is not negligible. Thus, $\gamma \neq 0$ and and so it is required to solve the entire equation (3.27). Thus, equation (3.27) takes the form

$$\ddot{W}_m(t) + \frac{\omega_1(m,k)}{\omega_0(m,k)}W_m(t) + \frac{\gamma}{\omega_1(m,k)}\Bigg[\bigg(\frac{D_1(m,k)}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n+1)\pi ct}{(2n+1)}D_2(n,m,k) - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n+1)\pi ct}{(2n+1)}D_3(n,m,k)\bigg)\ddot{W}_m(t) + 2c\bigg(\frac{D_4(m,k)}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n+1)\pi ct}{(2n+1)}D_5(n,m,k) - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n+1)\pi ct}{(2n+1)}D_6(n,m,k)\bigg)\dot{W}_m(t) + c^2\bigg(\frac{D_7(m,k)}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n+1)\pi ct}{(2n+1)}D_8(n,m,k) - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n+1)\pi ct}{(2n+1)}D_9(n,m,k)\bigg)W_m(t)\Bigg] = \frac{PL}{\mu_0\omega_0(m,k)\lambda_k}\Bigg[\beta_k + \cos\alpha_k - A_k\sin\alpha_k - B_k\cosh\alpha_k - C_k\sinh\alpha_k\Bigg] \tag{3.34}$$

On further re-arrangements, one obtains

$$\ddot{W}_m(t) + \frac{\frac{2c\gamma}{\omega_0(m,k)}\left(\frac{D_4(m,k)}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n+1)\pi ct}{(2n+1)}D_5(n,m,k) - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n+1)\pi ct}{(2n+1)}D_6(n,m,k)\right)}{1 + \frac{\gamma}{\omega_0(m,k)}\left(\frac{D_1(m,k)}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n+1)\pi ct}{(2n+1)}D_2(n,m,k) - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n+1)\pi ct}{(2n+1)}D_3(n,m,k)\right)}$$
$$\dot{W}_m(t) + \frac{\theta_m^2 + \left[\frac{c^2\gamma}{\omega_0(m,k)}\left(\frac{D_7(m,k)}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n+1)\pi ct}{(2n+1)}D_8(n,m,k) - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n+1)\pi ct}{(2n+1)}D_9(n,m,k)\right)\right]}{1 + \frac{\gamma}{\omega_0(m,k)}\left(\frac{D_1(m,k)}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n+1)\pi ct}{(2n+1)}D_2(n,m,k) - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n+1)\pi ct}{(2n+1)}D_3(n,m,k)\right)}]$$
$$W_m(t) = \frac{\frac{PL}{\mu_0\omega_0(m,k)\lambda_k}}{1 + \frac{\gamma}{\omega_0(m,k)}\left(\frac{D_1(m,k)}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n+1)\pi ct}{(2n+1)}D_2(n,m,k) - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n+1)\pi ct}{(2n+1)}D_3(n,m,k)\right)}$$
$$\Bigg[\beta_k + \cos\alpha_k - A_k\sin\alpha_k - B_k\cosh\alpha_k - C_k\sinh\alpha_k\Bigg] \tag{3.35}$$

Indisputably, unlike the moving force problem, an exact analytical solution to equation (3.35) is not possible. In order to obtain approximate analytical solution, one makes use of a modification of the asymptotic method of Struble. By this method, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the effect of the moving mass. An equivalent system operator defined by the modified frequency then replaces equation (3.35).
We shall consider a parameter $\gamma_0 < 1$ for any arbitrary mass ratio defined by

$$\gamma_0 = \frac{\gamma}{1+\gamma} \tag{3.36}$$

By using binomial theorem and truncating after second terms, one obtains

$$\gamma_0 = \gamma - 0(\gamma 2) \tag{3.37}$$

equation (3.37) becomes

$$\gamma_0 = \gamma \tag{3.38}$$

to $0(\gamma)$ only
and from equation (3.35)

$$\frac{1}{1+\frac{\gamma}{\omega_0(m,k)}(\frac{D_1(m,k)}{4}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n+1)\pi ct}{(2n+1)}D_2(n,m,k)-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n+1)\pi ct}{(2n+1)}} \tag{3.39}$$

becomes

$$[1-\frac{\gamma}{\omega_0(m,k)}(\frac{D_1(m,k)}{4}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n+1)\pi ct}{(2n+1)}D_2(n,m,k)-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n+1)\pi ct}{(2n+1)}D_3(n,m,k))] \tag{3.40}$$

$$|\frac{\gamma}{\omega_0(m,k)}(\frac{D_1(m,k)}{4}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n+1)\pi ct}{(2n+1)}D_2(n,m,k)-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n+1)\pi ct}{(2n+1)}D_3(n,m,k))|<1 \tag{3.41}$$

substituting equations (3.40) and (3.41) into equation (3.35), one obtains

$$\begin{aligned}&\ddot{W}_m(t)+2c\gamma_0\left(\frac{D_4(m,k)}{4}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n+1)\pi ct}{(2n+1)}D_5(n,m,k)-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n+1)\pi ct}{(2n+1)}D_6(n,m,k)\right)\\ &\dot{W}_m(t)+\left[\theta_m^2-\theta_m^2\gamma_0\left(\frac{\omega_1(m,k)}{\omega 0}+\left[\left(\frac{D_1(m,k)}{4}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n+1)\pi ct}{(2n+1)}D_2(n,m,k)-\right.\right.\right.\right.\\ &\left.\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n+1)\pi ct}{(2n+1)}\quad D_3(n,m,k)\right)+c^2\gamma_0\left(\frac{D_7(m,k)}{4}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n+1)\pi ct}{(2n+1)}D_8(n,m,k)-\right.\\ &\left.\left.\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n+1)\pi ct}{(2n+1)}\quad D_9(n,m,k)\right)\right]W_m(t)=\frac{PL}{\mu_0\omega_0(m,k)\lambda_k}\left[\beta_k+\cos\alpha_k-A_k\sin\alpha_k\right.\\ &\left.-B_k\cosh\alpha_k-C_k\sinh\alpha_k\right]\end{aligned} \tag{3.42}$$

to $0(\gamma_0)$ only

Applying method of Struble technique to equation (3.42), one obtains

$$\frac{d^2}{dt^2}W_m(t)+\theta_{mm}W_m(t)=\frac{PL}{\mu_0\omega_0(m,k)\lambda_k}\left[\beta_k+\cos\alpha_k-A_k\sin\alpha_k-B_k\cosh\alpha_k-C_k\sinh\alpha_k\right] \tag{3.43}$$

where

$$\theta_m m=\theta_m\left(1-\frac{\gamma_0}{2}\left[\frac{\theta_m D_1(m,k)}{4\omega_0(m,k)}-\frac{c^2D_7(m,k)}{4\theta_m^2\omega_0(m,k)}\right]\right) \tag{3.44}$$

solving equation (3.43) using Laplace transformation and convolution theory and taking into account equation (2.4), one obtains

$$V_m(x,t) = \frac{PL}{\mu_0\omega_0(m,k)\lambda_k} \times \frac{1}{\theta_{mm}^2(\theta_{mm}^4 - \alpha_k^4)}\Big[\beta_k(1-\cos\theta_{mm}t)(\theta_{mm}^4 - \alpha_k^4) + \theta_{mm}^2(\theta_{mm}^2 + \alpha_k^2)(\cos\alpha_k t - \cos\theta_{mm}t) - A_k\theta_{mm}(\theta_{mm}^2 + \alpha_k^2)(\theta_{mm}\sin\alpha_k t - \alpha_k\sin\theta_{mm}t) - B_k\theta_{mm}^2(\theta_{mm}^2 - \alpha_k^2)(\cosh\alpha_k t - \cos\theta_{mm}t) - C_k\theta_{mm}(\theta_{mm}^2 + \alpha_k^2)(\theta_{mm}\sinh\alpha_k t - \alpha_k\sin\theta_{mm}t)\Big] \times \left(\sin\frac{\lambda_m x}{L} + A_m\cos\frac{\lambda_m x}{L} + B_m\sinh\frac{\lambda_m x}{L} + C_m\cosh\frac{\lambda_m x}{L}\right) \tag{3.45}$$

equation (3.45) represents the transverse deflection of the non-uniform Rayleigh beam under moving distributed mass and resting on variable Pasternak elastic foundation.

# 4 Discussion of the Analytical Solutions

For this undamped system, it is desirable to examine the phenomenon of resonance. From equation (3.33), it is clearly shown that the beam resting on variable bi-parametric elastic foundation and traversed by a moving distributed force reaches a state of resonance whenever

$$\theta_m = \alpha_k \tag{4.1}$$

where

$$\alpha_k = \frac{\lambda_k c}{L} \tag{4.2}$$

that is

$$\theta_m = \frac{\lambda_k c}{L} \tag{4.3}$$

equation (3.45) shows that the same beam under the action of moving distributed mass experiences resonance effect whenever

$$\theta_{mm} = \frac{\lambda_k c}{L} \tag{4.4}$$

From equation (3.44)

$$\theta_m\left(1 - \frac{\gamma_0}{2}\left[\frac{\theta_m D_1(m,k)}{4\omega_0(m,k)} - \frac{c^2 D_7(m,k)}{4\theta_m^2\omega_0(m,k)}\right]\right) = \frac{\lambda_k c}{L} \tag{4.5}$$

It is therefore clear that for the same natural frequency, the critical speed for the system consisting of elastically supported non-uniform Rayleigh beam resting on variable elastic foundation and traverse by moving distributed force with uniform speed is greater than that of moving distributed mass problem. Thus,for the same natural frequency, resonance is reached faster in the moving distributed mass system than in the moving distributed force system.

# 5 Illustrative Examples

**a. Clamped Elastic Boundary Conditions**

At the clamped end, both the deflection and the slope vanish. Thus, when the Rayleigh beam is clamped at $x = 0$ and elastically supported at $x = L$, the conditions are expressed as

$$V(0,t) = 0 = V'(0,t) \tag{5.1}$$

at the end $x = 0$

$$V''(L,t) - k_1 V'(L,t) = 0 = V'''(L,t) - k_2 V(L,t) \quad (5.2)$$

at the end $x = L$
and for the normal modes

$$U_m(0) = 0 = U'_m(0) \quad (5.3)$$

at the end $x = 0$ and

$$U''_m(L) - k_1 U'_m(L) = 0 = U'''_m(L) - k_2 U_m(L) \quad (5.4)$$

at the end $x = L$
which implies that

$$U_k(0) = 0 = U'_k(0) \quad (5.5)$$

and

$$U''_k(L) - k_1 U'_k(L) = 0 = U'''_k(L) - k_2 U_k(L) \quad (5.6)$$

using equations (5.3) and (5.4), it can be shown that at $x = 0$ ,

$$A_m = -C_m, B_m = -1 \quad (5.7)$$

and at $x = L$ , using equation (5.6)

$$A_m = \frac{\frac{\lambda_m}{L}[\sin\lambda_m + \sinh\lambda_m] + k_1[\cos\lambda_m - \cosh\lambda_m]}{\frac{\lambda_m}{L}[\cos\lambda_m + \cosh\lambda_m] - k_1[\sin\lambda_m + \sinh\lambda_m]} = \frac{\frac{\lambda_m^3}{L^3}[\cos\lambda_m + \cosh\lambda_m] + k_2[\sinh\lambda_m - \sin\lambda_m]}{\frac{-\lambda_m^3}{L^3}[\sin\lambda_m - \sinh\lambda_m] + k_2[\cos\lambda_m - \cosh\lambda_m]}$$
$$= -C_m \quad (5.8)$$

From equation (5.8), one obtains

$$\tan\lambda_m = \tanh\lambda_m \quad (5.9)$$

Hence, we have

$$\lambda_1 = 3.927, \lambda_2 = 7.069, \lambda_3 = 10.21, ... \quad (5.10)$$

putting equations (5.7), (5.8) and (5.10) into equations (3.33) and (3.45), one obtains the displacement response respectively to a moving force and a moving mass of clamped elastic ends Rayleigh beam on a variable foundation.

**b. Elastically Supported Conditions at Both Ends**

For the case when the beam is elastically supported both at $x = 0$ and $x = L$, the conditions are expressed as

$$V''(0,t) - k_1 V'(0,t) = 0 = V'''(0,t) + k_2 V(0,t) \quad (5.11)$$

at $x = 0$ and

$$V''(L,t) - k_1 V'(L,t) = 0 = V'''(L,t) + k_2 V(L,t) \quad (5.12)$$

at $x = L$
Similarly, for normal modes

$$U''_m(0) - k_1 U'_m(0) = 0 = U'''_m(0) + k_2 U_m(0) \quad (5.13)$$

at $x = 0$ and

$$U''_m(L) - k_1 U'_m(L) = 0 = U'''_m(L) + k_2 U_m(L) \quad (5.14)$$

at $x = L$
which implies that

$$U''_k(0) - k_1 U'_k(0) = 0 = U'''_k(0) + k_2 U_k(0) \quad (5.15)$$

at $x = 0$ and

$$U_k^{''}(L) - k_1 U_k^{'}(L) = 0 = U_k^{'''}(L) + k_2 U_k(L) \tag{5.16}$$

at $x = L$
using equations (5.13) and (5.14), it can be shown that

$$C_m = \frac{[\frac{\lambda_m}{L} - k_1 r_2]\sin\lambda_m + [k_1 + \frac{r_2\lambda_m}{L}]\cos\lambda_m - \frac{r_1\lambda_m}{L}\sinh\lambda_m + k_1 r_1\cosh\lambda_m}{k_1 r_1\sin\lambda_m - \frac{r_1\lambda_m}{L}\cos\lambda_m + [\frac{r_3\lambda_m}{L} - k_1]\sinh\lambda_m + [\frac{\lambda_m}{L} - k_1 r_3]\cosh\lambda_m}$$
$$= -\frac{[\frac{r_2\lambda_m^3}{L^3} + k_2]\sin\lambda_m + [\frac{\lambda_m^3}{L^3} - k_2 r_2]\cos\lambda_m - k_2 r_1\sinh\lambda_m - \frac{r_1\lambda_m^3}{L^3}\cosh\lambda_m}{\frac{r_1\lambda_m^3}{L^3}\sin\lambda_m + k_2 r_1\cos\lambda_m + [\frac{\lambda_m^3}{L^3} + k_2 r_3]\sinh\lambda_m + [\frac{r_3\lambda_m^3}{L^3} + k_2]\cosh\lambda_m} \tag{5.17}$$

$$A_m = r_1 C_m + r_2, B_m = r_3 C_m + r_1 \tag{5.18}$$

where

$$r_1 = \frac{\frac{\lambda_m^4}{L^4} + k_1 k_2}{\frac{\lambda_m^4}{L^4} - k_1 k_2}, r_2 = \frac{-\frac{2k_1\lambda_m^3}{L^3}}{\frac{\lambda_m^4}{L^4} - k_1 k_2}, r_3 = \frac{-\frac{2k_1\lambda_m}{L}}{\frac{\lambda_m^4}{L^4} - k_1 k_2} \tag{5.19}$$

using equations (5.17), (5.18) and (5.19), the modified frequency equation for the dynamical problem is obtained as

$$\tan\lambda_m = \tanh\lambda_m \tag{5.20}$$

Hence, we have

$$\lambda_1 = 3.927, \lambda_2 = 7.069, \lambda_3 = 10.21, ... \tag{5.21}$$

substituting equations (5.17), (5.18), (5.19) and (5.20) into equations (3.33) and (3.45), one obtains the displacement response respectively to a moving force and a moving mass of Rayleigh beam elastically supported at both ends on a variable foundation.

# 6 Numerical Results and Discussions

To illustrate the analysis presented in this work, the non-uniform Rayleigh beam is taken to be of length $L = 12.192m$, the load velocity $c = 8.128m/s$ and modulus of elasticity $E_0 = 2.109 \times 10^9 kg/m$, the moment of inertia $I_0 = 2.37698 \times 10^{-3} m^4$.

**a. Graphs for Clamped-Elastic Boundary Conditions**

Figs 1 and 2 display the effect of foundation modulus $S_0$ on the deflection profile of clamped elastic Rayleigh beam under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitudes decrease as the value of $S_0$ increases.

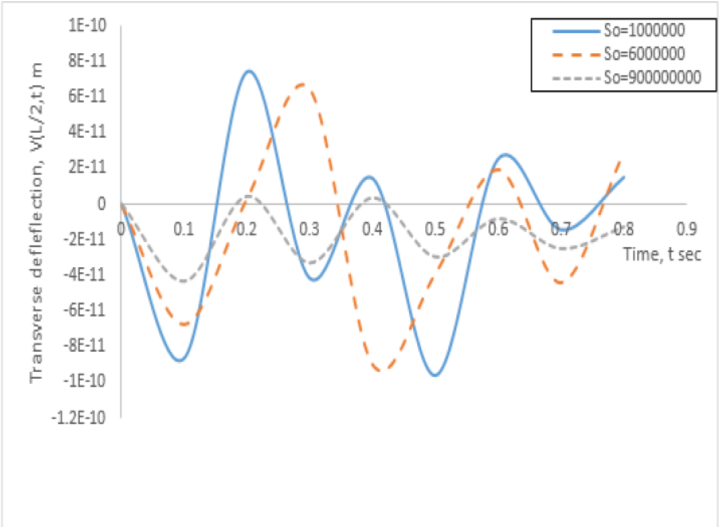


**Fig. 1. Deflection profile of a clamped elastic non-uniform Rayleigh beam on variable foundation and traversed by moving distributed force for values of $R_0 = 30$, $N_0 = 20000$, $K_0 = 20000000$ and various values of $S_0$**

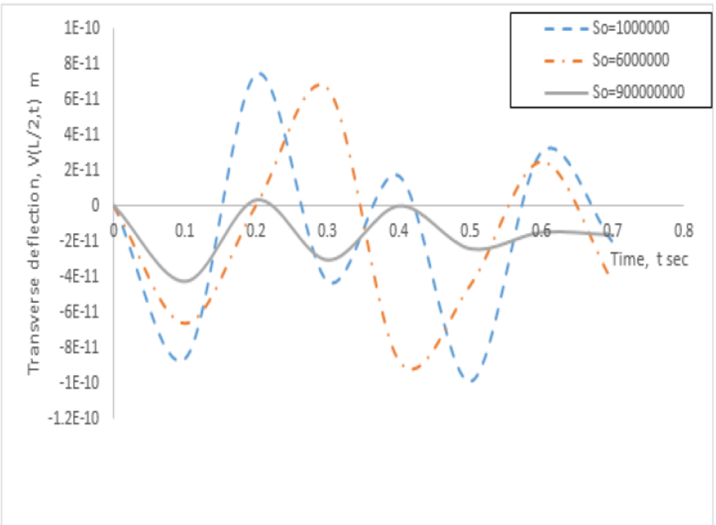


**Fig. 2. Deflection profile of a clamped elastic non-uniform Rayleigh beam on variable foundation and traversed by moving distributed mass for values of $R_0 = 30$, $N_0 = 20000$, $K_0 = 20000000$ and various values of $S_0$**

Figs 3 and 4 display the effect of shear modulus $K_0$ on the deflection profile of clamped elastic Rayleigh beam under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitudes decrease as the value of $K_0$ increases.

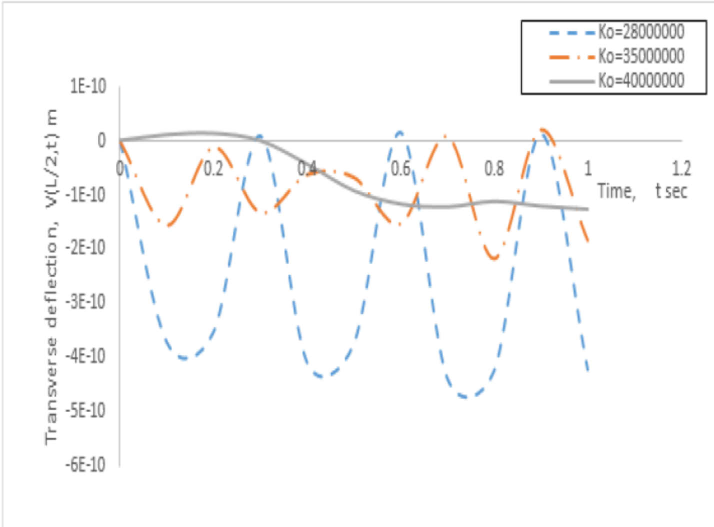


**Fig. 3. Deflection profile of a clamped elastic non-uniform Rayleigh beam on variable foundation and traversed by moving distributed force for values of $R_0 = 30$, $N_0 = 20000$, $S_0 = 300000000$ and various values of $K_0$**

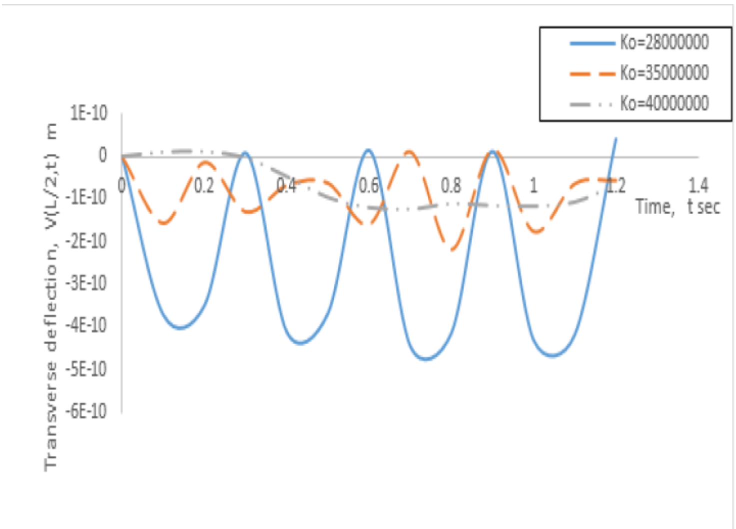


**Fig. 4. Deflection profile of a clamped elastic non-uniform Rayleigh beam on variable foundation and traversed by moving distributed mass for values of $R_0 = 30$, $N_0 = 20000$, $S_0 = 300000000$ and various values of $K_0$**

Figs 5 and 6 display the effect of rotatory inertia $R_0$ on the deflection profile of clamped elastic Rayleigh beam under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitudes decrease as the value of $R_0$ increases.

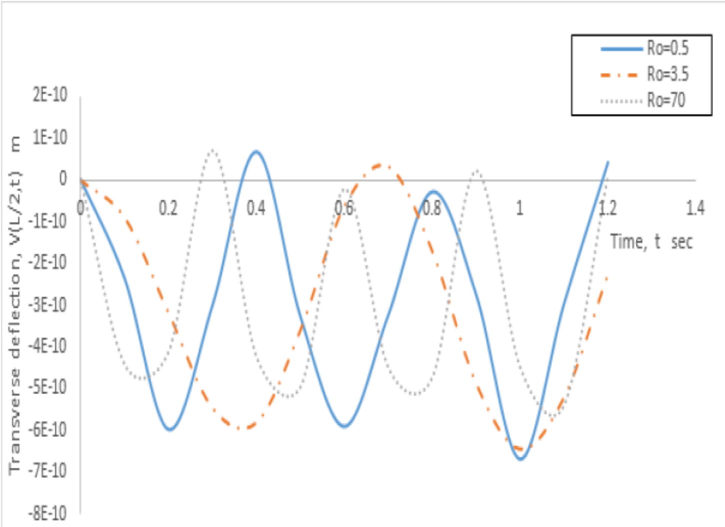


**Fig. 5. Deflection profile of a clamped elastic non-uniform Rayleigh beam on variable foundation and traversed by moving distributed force for values of $K_0 = 20000000$, $N_0 = 20000$, $S_0 = 300000000$ and various values of $R_0$**

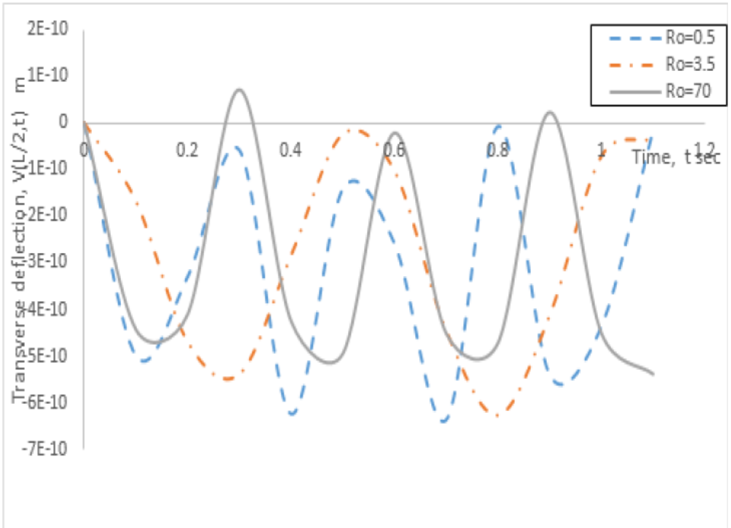


**Fig. 6. Deflection profile of a clamped elastic non-uniform Rayleigh beam on variable foundation and traversed by moving distributed mass for values of $K_0 = 20000000$, $N_0 = 20000$, $S_0 = 300000000$ and various values of $R_0$**

For the purpose of comparison, Fig 7 shows the displacement curves of moving distributed force and moving distributed mass for fixed values of rotatory inertia, foundation modulus and shear modulus for clamped elastic boundary conditions. The graph shows that the response amplitude of a moving mass is greater than that of a moving force problem.

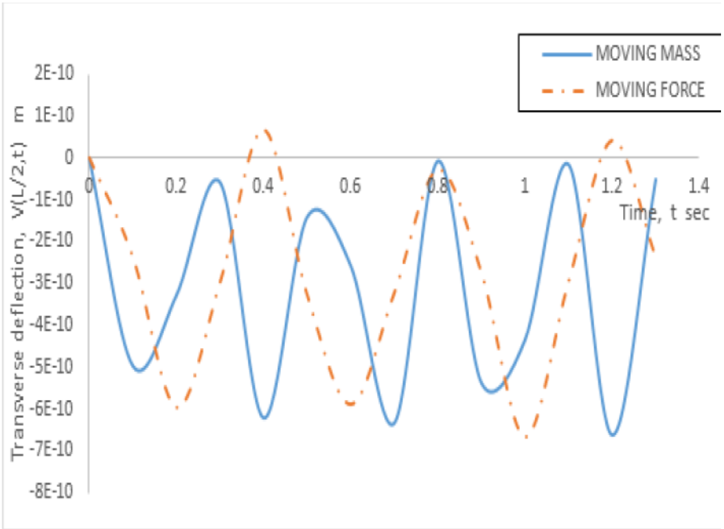


**Fig. 7. Comparison of the deflections of moving distributed force and moving distributed mass for clamped elastic boundary conditions**

**b. Graphs for Elastic-Elastic Boundary Conditions**

Figs 8 and 9 display the effect of foundation modulus $S_0$ on the deflection profile of elastic-elastic Rayleigh beam under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitudes decrease as the value of $S_0$ increases.

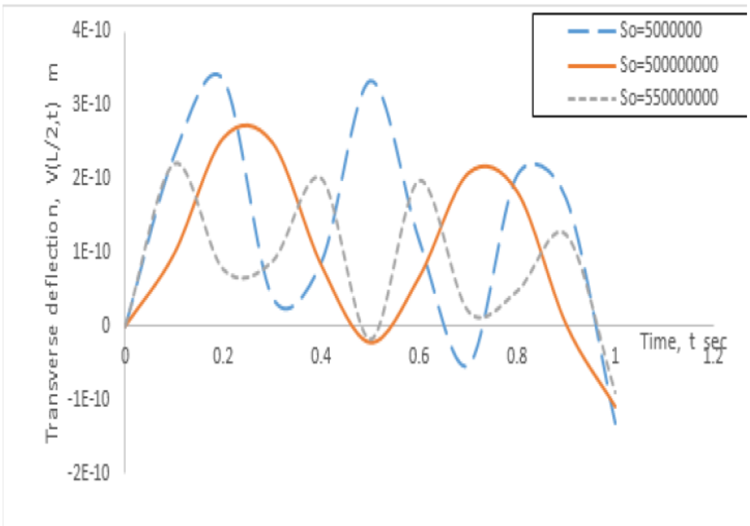


**Fig. 8. Deflection profile of an elastic-elastic non-uniform Rayleigh beam on variable foundation and traversed by moving distributed force for values of $R_0 = 0$, $N_0 = 20000$, $K_0 = 20000000$ and various values of $S_0$**

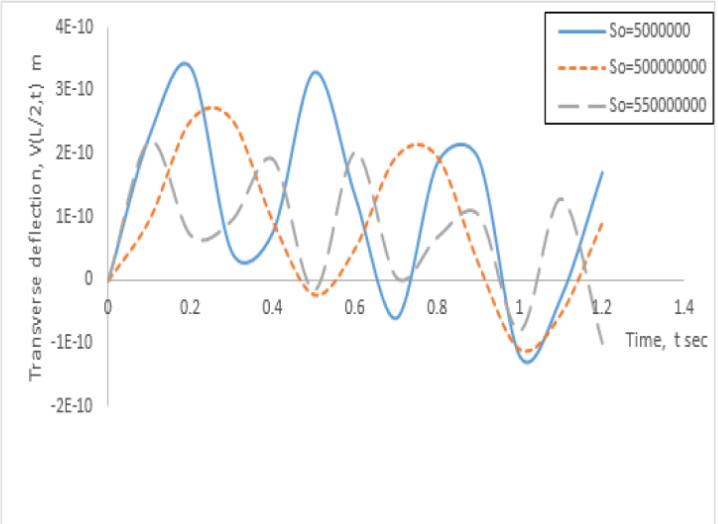


**Fig. 9. Deflection profile of an elastic-elastic non-uniform Rayleigh beam on variable foundation and traversed by moving distributed mass for values of $R_0 = 0$, $N_0 = 20000$, $K_0 = 20000000$ and various values of $S_0$**

Figs. 10 and 11 display the effect of shear modulus $K_0$ on the deflection profile of elastic-elastic Rayleigh beam under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitudes decrease as the value of $K_0$ increases.

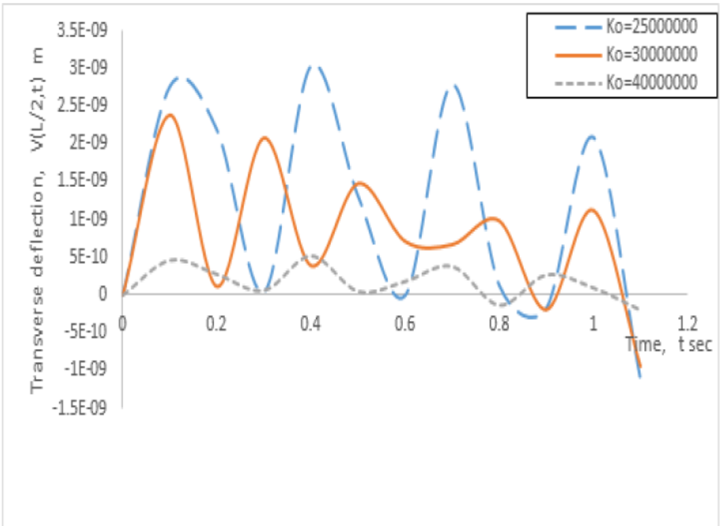


**Fig. 10. Deflection profile of an elastic-elastic non-uniform Rayleigh beam on variable foundation and traversed by moving distributed force for values of $R_0 = 30$, $N_0 = 20000$, $S_0 = 300000000$ and various values of $K_0$**

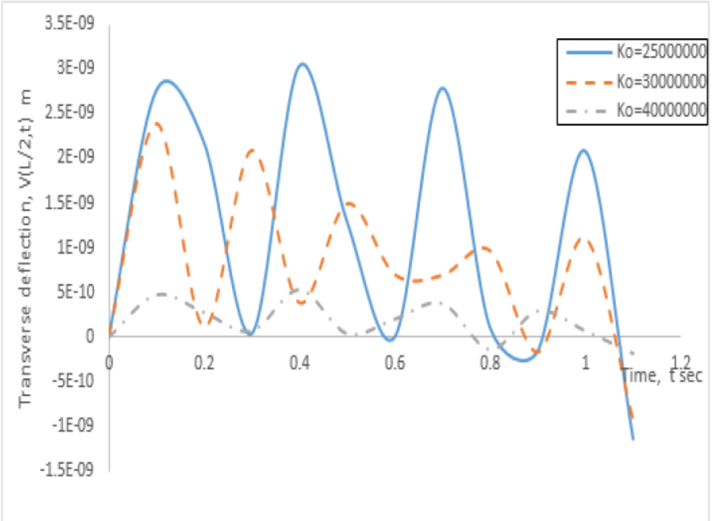


**Fig. 11. Deflection profile of an elastic-elastic non-uniform Rayleigh beam on variable foundation and traversed by moving distributed mass for values of $R_0 = 30$, $N_0 = 20000$, $S_0 = 300000000$ and various values of $K_0$**

Figs. 12 and 13 display the effect of rotatory inertia $R_0$ on the deflection profile of elastic-elastic Rayleigh beam under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitudes decrease as the value of $R_0$ increases.

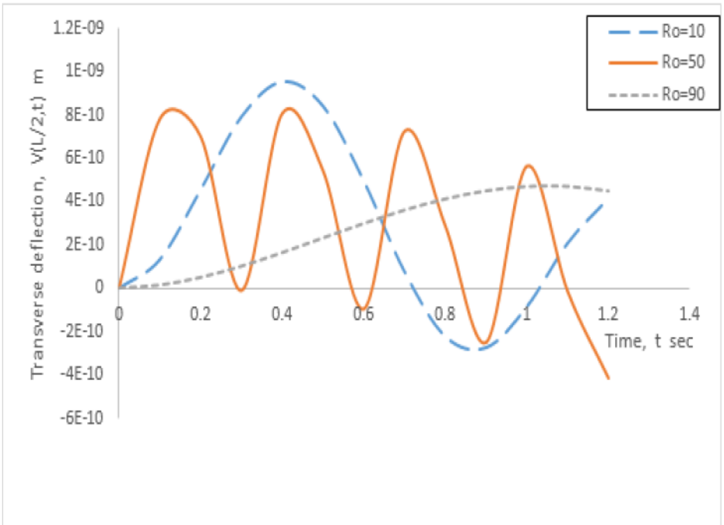


**Fig. 12. Deflection profile of an elastic-elastic non-uniform Rayleigh beam on variable foundation and traversed by moving distributed force for values of $K_0 = 20000000$, $N_0 = 20000$, $S_0 = 300000000$ and various values of $R_0$**

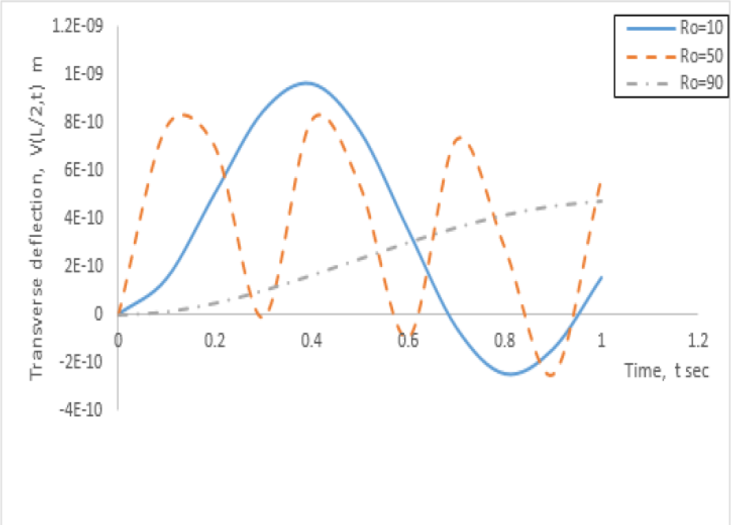


**Fig. 13. Deflection profile of an elastic-elastic non-uniform Rayleigh beam on variable foundation and traversed by moving distributed mass for values of $K_0 = 20000000$, $N_0 = 20000$, $S_0 = 300000000$ and various values of $R_0$**

For the purpose of comparison, Fig. 14 shows the displacement curves of moving distributed force and moving distributed mass for fixed values of rotatory inertia, foundation modulus and shear modulus for elastic-elastic boundary conditions. The graph shows that the response amplitude of a moving mass is greater than that of a moving force problem.

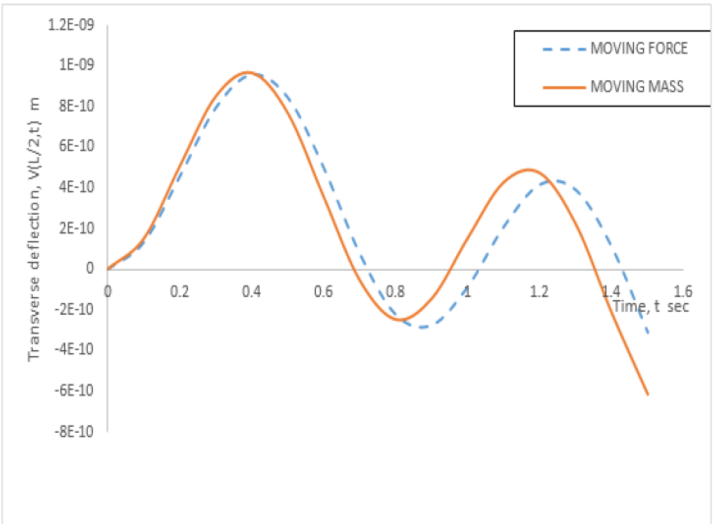


**Fig. 14. Comparison of the deflections of moving distributed force and moving distributed mass for elastic-elastic boundary conditions**

# 7 Conclusion

In this research work, the problem concerning the influence of rotatory inertia correction factor on the vibration of elastically supported non uniform Rayleigh beam under moving distributed mass and resting on bi-parametric elastic foundation has been studied. The closed form solutions of the fourth order partial differential equations with variable and singular co-efficients are obtained for both cases of moving force and moving mass. The solutions are analyzed and resonance conditions are obtained for the problem. The results in plotted curves show the influence of rotatory inertial correction factor as well as the shear modulus and foundation modulus on the beam for both cases of moving distributed force and moving distributed mass.

# Competing Interests

Authors have declared that no competing interests exist.

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