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# Three- and Four-Dimensional Generalized Pythagorean Numbers

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## Abstract

The Pythagorean triples $(a,b\,|\,c)$ of planar geometry which satisfy the equation $a^2+b^2=c^2$ with integers $(a,b,c)$ are generalized to 3D-Pythagorean quadruples $(a,b,c\,|\,d)$ of spatial geometry which satisfy the equation $a^2+b^2+c^2=d^2$ with integers $(a,b,c,d)$. Rules for a parametrization of the numbers $(a,b,c,d)$ are derived and a list of all possible nonequivalent cases without common divisors up to $d^2<1000$ is established. The 3D-Pythagorean quadruples are then generalized to 4D-Pythagorean quintuples $(a,b,c,d\,|\,e)$ which satisfy the equation $a^2+b^2+c^2+d^2=e^2$ and a parametrization is derived. Relations to the 4-square identity are discussed which leads also to the $N$-dimensional case. The initial 3D- and 4D-Pythagorean numbers are explicitly calculated up to $d^2<1000$, respectively, $e^2<500$.

## Keywords

Number Theory, Pythagorean Triples, Tesseract, 4-Square Identity, Diophantine Equation

## 1. Introduction

Pythagorean triples are triples of integers $(a,b\,|\,c)$ which satisfy the equality (e.g., Stillwell [1] [2], Rademacher and Toeplitz [3])

$$\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}\right)^2=1. \tag{1.1}$$

In plane geometry of right triangles with side lengths $(a,b\,|\,c)$ ($a,b$ catheti or legs and $c$ hypotenuse), this means that they satisfy the theorem of Pythagoras

$$a^2+b^2=c^2. \tag{1.2}$$

with integers $(a,b\,|\,c)$ with the best-known special case $(3,4\,|\,5)$. One has here to mention that this theorem which is now connected with the name of Pythagoras was known in different parts of the Old World already before the life of the great Greek philosopher (e.g., Stillwell [4], Pickover [5]).

Definition (1.1) shows that integer multiples of these numbers are trivial cases which can be excluded if one wants to consider the basically different triples of such numbers. The basically different doubles $\left(\frac{a}{c},\frac{b}{c}\right)$ can be also considered as coordinates of unit vectors in the first quadrant of the unit circle with positive rational coordinates $\left(\frac{a}{c},\frac{b}{c}\right)$. The book of Beiler [6] discusses some more problems which arise within the theory of Pythagorean triples and presents interesting series of such triples under different aspects as in most other presentations on this topic.

The most famous generalization of the Pythagorean triples is Fermat's Last Theorem (e.g., [3] [4] [5]) which states that the Diophantine equation

$$a^n+b^n=c^n, \quad (n>2), \tag{1.3}$$

does not possess positive integer solutions for $n\geq 3$ and which in the opinion of the world's whole mathematical community was completely proved only in recent times 1993-1994 by Andrew Wiles. This is written in many articles and books, in particular, also in popular representations (e.g., Singh [7]).

The present paper concerns the three- and four-dimensional generalizations of Pythagorean triples to that what I will call 3D-Pythagorean quadruples $(a,b,c\,|\,d)$ and 4D-Pythagorean quintuples $(a,b,c,d\,|\,e)$. This seems to be new. The 3D-Pythagorean quadruples $(a,b,c\,|\,d)$, by definition, should satisfy the equation

$$\left(\frac{a}{d}\right)^2+\left(\frac{b}{d}\right)^2+\left(\frac{c}{d}\right)^2=1, \tag{1.4}$$

with rational triples $\left(\frac{a}{d},\frac{b}{d},\frac{c}{d}\right)$. In such form, they can be considered as the coordinates of unit vectors with positive rational coordinates in the spatial angle $\frac{\pi}{2}$ (the first 8th part of the full spatial angle). In the form

$$a^2+b^2+c^2=d^2, \tag{1.5}$$

they are the side lengths $(a,b,c)$ of a cuboid with spatial diagonal $d$ and with integer numbers $(a,b,c,d)$ and the basic such numbers should not possess a common divisor. Since this concerns our three-dimensional space it is directly accessible to our view.

The 4D-Pythagorean quintuples $(a,b,c,d\,|\,e)$ satisfy the equation

$$\left(\frac{a}{e}\right)^2+\left(\frac{b}{e}\right)^2+\left(\frac{c}{e}\right)^2+\left(\frac{d}{e}\right)^2=1, \tag{1.6}$$

with rational quadruples $\left(\frac{a}{e},\frac{b}{e},\frac{c}{e},\frac{d}{e}\right)$ or the equation

$$a^2+b^2+c^2+d^2=e^2, \tag{1.7}$$

with integer quintuples $(a,b,c,d\,|\,e)$. In the last form, they may be considered as the four-dimensional analogue of three-dimensional cuboids where $e$ is the four-dimensional spatial diagonal. Although they are no more accessible to our direct views, mathematics has developed means to get a partial imagination, in this case, by projection of this cuboid onto the three- or even two-dimensional space. The four-dimensional cuboid with equal side lengths $a=b=c=d$ is called a tesseract. A beautiful picture of its projection onto the paper plane gives also a good imagination of a three-dimensional projection of the four-dimensional tesseract we may find in the book of Pickover [5] (p. 283) and it seems to be not difficult to imagine this projection stretched with different lengths of the edges.

Algebraically, it does not make it difficult to consider generalizations of the Pythagorean triples to higher dimensions ($N>4$) and the possibilities to find such $(N+1)$-tuples increase with increasing $N$ considerably.

## 2. 3D-Pythagorean Quadruples $(a,b,c\,|\,d)$ and a Parametrization

In this Section, we try to find out some simple rules for the basic quadruples $(a,b,c\,|\,d)$ in (1.5) without a common divisor and try to establish the initial such quadruples up to $d^2<1000$.

First, we easily find that $d$ cannot be an even number $d=2p,(p=1,2,\cdots)$. Assuming that $d$ is an even number implies $d^2=4p^2$ has to be divisible by 4. In this case, we have two possibilities for $(a,b,c)$: first, $(a=2l,b=2m,c=2n)$ are all even numbers and then $(a,b,c,d)$ possess the common divisor 2 that was excluded for basic quadruples; second, one of the numbers $(a,b,c)$ is an even number and two are odd numbers, say $(a=2l,b=2m+1,c=2n+1)$. Then

$$a^2+b^2+c^2=4\left(l^2+m^2+n^2+m+n\right)+2=4p^2=d^2. \tag{2.1}$$

The left-hand side is not divisible by 4 and thus the case of even $d=2p$ is impossible. Therefore, $d$ has to be an odd number that we now assume.

For an odd number $d=2p+1,(p=0,1,2,\cdots)$ we have two possibilities for $(a,b,c)$: First, $(a=2l+1,b=2m+1,c=2n+1)$ are only odd numbers $(a=2l+1,b=2m+1,c=2n+1)$ with

$$a^2+b^2+c^2=4\left(l^2+m^2+n^2+l+m+n\right)+3=4\left(p^2+p\right)+1=d^2, \tag{2.2}$$

that is impossible since the remainder of division by 4 is different with both sides; The second one of the three numbers $(a,b,c)$ is an odd number and the other two are even numbers, say $(a=2l+1,b=2m,c=2n)$. Then we have

$$a^2+b^2+c^2=4\left(l^2+m^2+n^2+l\right)+1=4\left(p^2+p\right)+1=d^2, \tag{2.3}$$

and both sides possess the same remainder by division with 4. Thus the only possible basic cases are where one of the numbers $(a,b,c)$ is an odd number and the other two are even numbers and that $d$ is an odd number that means

$$(a,b,c\,|\,d)=(2l+1,2m,2n\,|\,2p+1),\quad (p=1,2,3,\cdots). \tag{2.4}$$

Inserting the parametrization of the right-hand side of (2.4) into the equality (1.5) leads after division by 4 to the following equality

$$m^2+n^2=(p-l)(p+l+1). \tag{2.5}$$

The product $(p-l)(p+l+1)$ is in every case an even number since if $p-l$ is an even number then $p+l+1$ is an odd number and if $p-l$ is an odd number then $p+l+1$ is an even number. This means that the left-hand side $m^2+n^2$ is an even number that admits the two possibilities that $m$ and $n$ are both even numbers or both odd numbers.

## 3. Basic 3D-Pythagorean Quadruples $(a,b,c\,|\,d)$ up to $d^2<1000$

Making an arbitrary of the 6 possible permutations of $(a,b,c)$ leads to the same $d$ and therefore these permutations are equivalent. To list out only the nonequivalent triples $(a,b,c)$ one may make an ordering of these triples.

By computer we found the following nonequivalent basic quadruples $(a,b,c\,|\,d)$ first ordered with increasing numbers $d$ and then in the new ordering $a\le b\le c$) which satisfy relation (2.1) up to $d^2<1000$, ($d\le 31$).

$$\begin{aligned}
&(1,2,2\,|\,3),\quad (2,3,6\,|\,7),\quad (1,4,8\,|\,9),\quad (4,4,7\,|\,9),\quad (2,6,9\,|\,11),\\
&(6,6,7\,|\,11),\quad (3,4,12\,|\,13),\quad (2,5,14\,|\,15),\quad (2,10,11\,|\,15),\\
&(1,12,12\,|\,17),\quad (8,9,12\,|\,17),\quad (1,6,18\,|\,19),\quad (6,6,17\,|\,19),\\
&(6,10,15\,|\,19),\quad (4,5,20\,|\,21),\quad (4,8,19\,|\,21),\quad (4,13,16\,|\,21),\\
&(8,11,16\,|\,21),\quad (3,6,22\,|\,23),\quad (3,14,18\,|\,23),\quad (6,13,18\,|\,23),\\
&(9,12,20\,|\,25),\quad (12,15,16\,|\,25),\quad (2,7,26\,|\,27),\quad (2,10,25\,|\,27),\\
&(2,14,23\,|\,27),\quad (7,14,22\,|\,27),\quad (10,10,23\,|\,27),\quad (3,16,24\,|\,29),\\
&(11,12,24\,|\,29),\quad (12,16,21\,|\,29),\quad (5,6,30\,|\,31),\quad (6,14,27\,|\,31),\\
&(6,21,22\,|\,31),\quad (14,18,21\,|\,31).
\end{aligned} \tag{3.1}$$

Within the written list are cases where not only the spatial diagonal of the corresponding cuboid is an integer but also a facial diagonal. This happens for the cases $(3,4,12\,|\,13)$, $(8,9,12\,|\,17)$, $(12,15,16\,|\,25)$ and $(12,16,21\,|\,29)$ of the 3D-Pythagorean quadruples $(a,b,c\,|\,d)$ which possess an intersection with the simplest (2D)-Pythagorean triple $(3,4\,|\,5)$ or a multiple of it. Here we may pose the problem of whether or not are there such cases with intersections of two Pythagorean triples.

Beginning from $d=7$ every odd number $d=2p+1$ up to $d=31$ is present in the given lists. A problem is whether or not this is true also for all higher odd $d$. If there would be found a counter-example where this is not true then this is a disproof. However, in case that there will be not found a counter-example the proof that such an example does not exist seems to be difficult.

Up to now, we have not found a procedure to produce successively all basic quadruples $(a,b,c \mid d)$ with minimal expenditure that means without filtering them from numbers $d$ which according to (2.1) are not integers.

## 4. A Nearer Look to the Parametrization of the 3D-Pythagorean Quadruples $(a,b,c \mid d)$

In this Section we take a nearer look onto the parametrization (2.4) of the variables in (2.1) and use for this the lists (3.1) of explicitly determined numbers $(a,b,c,d)$ but with $a = 2l + 1$ the odd number and $(b = 2m, c = 2n)$ the even numbers. This leads to the following **Table 1**.

**Table 1.** 3D-Pythagorean quadruples $(a,b,c \mid d)$ and their relation to parameters $(l,m,n)$ and $p$.

| $d$ | $a$ | $b$ | $c$ | $p$ | $l$ | $m$ | $n$ | $p-l$ | $p+l+1$ | $n-m$ | $n+m$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 2 | 2 | 1 | 0 | 1 | 1 | 1 | 2 | 0 | 2 |
| 7 | 3 | 2 | 6 | 3 | 1 | 1 | 3 | 2 | 5 | 2 | 4 |
| 9 | 1 | 4 | 8 | 4 | 0 | 2 | 4 | 4 | 5 | 2 | 6 |
| 9 | 7 | 4 | 4 | 4 | 3 | 2 | 2 | 1 | 8 | 0 | 4 |
| 11 | 9 | 2 | 6 | 5 | 4 | 1 | 3 | 1 | 10 | 2 | 4 |
| 11 | 7 | 6 | 6 | 5 | 3 | 3 | 3 | 2 | 9 | 0 | 6 |
| 13 | 3 | 4 | 12 | 6 | 1 | 2 | 6 | 5 | 8 | 4 | 8 |
| 15 | 5 | 2 | 14 | 7 | 2 | 1 | 7 | 5 | 10 | 6 | 8 |
| 15 | 11 | 2 | 10 | 7 | 5 | 1 | 5 | 2 | 13 | 4 | 6 |
| 17 | 1 | 12 | 12 | 8 | 0 | 6 | 6 | 8 | 9 | 0 | 12 |
| 17 | 9 | 8 | 12 | 8 | 4 | 4 | 6 | 4 | 13 | 2 | 10 |
| 19 | 1 | 6 | 18 | 9 | 0 | 3 | 9 | 9 | 10 | 6 | 12 |
| 19 | 17 | 6 | 6 | 9 | 8 | 3 | 3 | 1 | 18 | 0 | 6 |
| 19 | 15 | 6 | 10 | 9 | 7 | 3 | 5 | 2 | 17 | 2 | 8 |
| 21 | 5 | 4 | 20 | 10 | 2 | 2 | 2 | 8 | 13 | 8 | 12 |
| 21 | 19 | 4 | 8 | 10 | 9 | 2 | 4 | 1 | 20 | 2 | 6 |
| 21 | 13 | 4 | 16 | 10 | 6 | 2 | 8 | 4 | 17 | 6 | 10 |
| 21 | 11 | 8 | 16 | 10 | 5 | 4 | 8 | 5 | 16 | 4 | 12 |
| 23 | 3 | 6 | 22 | 11 | 1 | 3 | 11 | 10 | 13 | 8 | 14 |
| 23 | 3 | 14 | 18 | 11 | 1 | 7 | 9 | 10 | 13 | 2 | 16 |
| 23 | 13 | 6 | 18 | 11 | 6 | 3 | 9 | 5 | 18 | 6 | 12 |
| 25 | 9 | 12 | 20 | 12 | 4 | 6 | 10 | 8 | 17 | 4 | 16 |
| 25 | 15 | 12 | 16 | 12 | 7 | 6 | 8 | 5 | 20 | 2 | 14 |
| 27 | 7 | 2 | 26 | 13 | 3 | 1 | 13 | 10 | 17 | 12 | 14 |
| 27 | 25 | 2 | 10 | 13 | 12 | 1 | 5 | 1 | 26 | 4 | 6 |
| 27 | 23 | 2 | 14 | 13 | 11 | 1 | 7 | 2 | 25 | 6 | 8 |
| 27 | 7 | 14 | 22 | 13 | 3 | 7 | 11 | 10 | 17 | 4 | 18 |
| 27 | 23 | 10 | 10 | 13 | 11 | 5 | 5 | 2 | 25 | 0 | 10 |
| 29 | 3 | 16 | 24 | 14 | 1 | 8 | 12 | 13 | 16 | 4 | 20 |

We see here explicitly that $m$ and $n$ are both even or both odd numbers and therefore $n-m$ and $n+m$ are in every case even numbers and that one of the two numbers $p-l$ and $p+l+1$ has to be an even number and the other one then an odd number and that both possibilities are realized. The condition (2.5) is easy to check in this table. However, it is not a minimal procedure which provides all basic nonequivalent quadruples $(a,b,c\,|\,d)$ in a systematic way as this is possible in the case of the usual (2D)-Pythagorean triples. Therefore, one cannot be sure to have overlooked a possible quadruple $(a,b,c\,|\,d)$ up to a certain maximal value $d$. A parametrization which provides only genuine quadruples follows from the 4-square identity which is generally applicable but does not immediately lead to a "best" ordering.

## 5. A Possible Generalization of 3D-Pythagorean Quadruples $(a,b,c\,|\,d)$ to Cubics and Quartics of Quadruples

As in the case of Pythagorean triples, there are different possibilities for generalization of the 3D-Pythagorean quadruples $(a,b,c\,|\,d)$. One such possibility is to look for integer positive solutions $(a,b,c,d)$ to the equation

$$a^n+b^n+c^n=d^n, \quad (n>2). \tag{4.1}$$

This is in analogy to the search for integer positive solutions of Equation (1.3) with the statement of the Last Fermat's problem finally proved. However, different from the nonexistence of such solutions in the case of Last Fermat's problem it is easy to show by examples that such solutions for (4.1) exist. Three examples for $n=3$ in (4.1) are

$$(3,4,5\,\|\,6), \quad (1,6,8\,\|\,9), \quad (3,10,18\,\|\,19). \tag{4.2}$$

It is very remarkable that the first such quadruple of cubics $(3,4,5\,\|\,6)$ (e.g., Beiler [6], p. 152) is in beautiful analogy to the first Pythagorean triple $(3,4\,|\,5)$

$$3^2+4^2=5^2, \quad 3^3+4^3+5^3=6^3. \tag{4.3}$$

However, if one expects that this could be continued to quintuples of quartics $(a,b,c,d\,\|\,e)$ and so on then this is wrong as the following Table 2 shows.

For $n=4$ in (4.1) we could not find an example and here arises the problem of giving proof of whether or not this is possible. For $n\geq 5$ we did not make investigations.

## 6. Generalization of 3D-Pythagorean Quadruples $(a,b,c\,|\,d)$ by Dimension to 4D-Pythagorean Quintuples $(a,b,c,d\,|\,e)$

In this Section we consider the generalization of 3D-Pythagorean quadruples

**Table 2.** Generalization of (4.3) by "Degree $n$ = Dimension $N$" leading to irrationals.

| Degree $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\left(\sum_{p=3}^{n+2} p^n\right)^{\frac{1}{n}}$ | 3 | 5 | 6 | 6.89336 | 7.80557 |

$(a,b,c\,|\,d)$ by dimension to 4D-Pythagorean quintuples $(a,b,c,d\,|\,e)$ which are integer solutions $(a,b,c,d,e)$ of the Diophantine equation

$$a^2+b^2+c^2+d^2=e^2. \tag{5.1}$$

This case is also interesting, in particular, because there exists a 4-square identity (e.g., Stillwell [4], Chapter 20.4, Conway and Smith [8], Chapter 6.10) derived from the algebra of quaternions. In **Appendix** we give a widely coordinate-invariant derivation of this 4-square identity. The problem is whether or not this leads to a parametrization of all possible 4D-Pythagorean quintuples $(a,b,c,d\,|\,e)$. We will show that it leads to a parametrization of all possible 4D-Pythagorean quintuples $(a,b,c,d\,|\,e)$ which, however, is not unique and may possess irrational parameters.

We analyze case (5.1) in analogy to case (1.5) where we are interested in finding the basic quintuples without a common divisor. If $e=2p$ is an even number then we have 3 possible cases for $(a,b,c,d)$. The first case is then that $(a=2k,b=2l,c=2m,d=2n)$ are all even numbers. Then $(a,b,c,d\,|\,e)$ possesses the common divisor 2 and this case can be excluded from the basic cases. The second case is that $(a=2k+1,b=2l+1,c=2m,d=2n)$ that means that two of the numbers $(a,b,c,d)$ are odd and two are even numbers from which follows

$$a^2+b^2+c^2+d^2=4\left(k^2+l^2+m^2+n^2+k+l\right)+2=4p^2=e^2. \tag{5.2}$$

The right-hand side is divisible by 4 but the left-hand side is not divisible by 4 and this case can be also excluded.

There remains the case $(a=2k+1,b=2l+1,c=2m+1,d=2n+1)$ that all numbers $(a,b,c,d)$ are odd numbers from which follows

$$a^2+b^2+c^2+d^2=4\left(k^2+l^2+m^2+n^2+k+l+m+n+1\right)=4p^2=e^2. \tag{5.3}$$

Both sides are here divisible by 4 and this is the only case for an even number $e=2p$. The first initial examples (6.1) of this case in Section 7 suggest that the possible numbers $p$ are furthermore restricted to $p=2q+1$ or $e=2(2q+1)$ with $q=0,1,2,\cdots$ as possible candidates. The proof can be made by representing (5.3) in the following way. The number $p$ cannot be an even number $p=2q$ since then the content in braces in

$$\begin{aligned} a^2+b^2+c^2+d^2 &= 4\left\{2\left(\underbrace{\frac{k(k+1)}{2}+\frac{l(l+1)}{2}+\frac{m(m+1)}{2}+\frac{n(n+1)}{2}}_{\text{sum of 4 integers}}\right)+1\right\} \\ &= 4\left\{4q^2\right\}=e^2, \end{aligned} \tag{5.4}$$

is an odd number on the left-hand side and an even number $4q^2$ on the right-hand side. Therefore $p$ has to be an odd number $p=2q+1$ for which we find from (5.3)

$$\begin{aligned} a^2+b^2+c^2+d^2 &= 4\left\{2\left(\frac{k(k+1)}{2}+\frac{l(l+1)}{2}+\frac{m(m+1)}{2}+\frac{n(n+1)}{2}\right)+1\right\} \\ &= 4(2q+1)^2=4\left\{4q(q+1)+1\right\}=e^2. \end{aligned} \tag{5.5}$$

From this we find the necessary equality

$$\frac{k(k+1)}{2}+\frac{l(l+1)}{2}+\frac{m(m+1)}{2}+\frac{n(n+1)}{2}=2\frac{q(q+1)}{2}, \qquad (5.6)$$

with $\frac{q(q+1)}{2}=0,1,3,6,10,15,\cdots$ as possible values. The combinations of $(k,l,m,n)$ which satisfy (5.6) for one of the possible $q$ are then 4D-Pythagorean quintuples.

We now consider the case $e=2p+1$ where $e$ is an odd number. Then there remains only the case $(a=2k+1, b=2l, c=2m, d=2n)$

$$a^2+b^2+c^2+d^2=4\left(k^2+l^2+m^2+n^2+k\right)+1=4\left(p^2+p\right)+1=e^2, \qquad (5.7)$$

and the remainder by division of both sides is the same and therefore this case is possible. The other case with 3 odd numbers and 1 even number $(a=2k+1, b=2l+1, c=2m+1, d=2n)$ is impossible since the remainder by division of both sides by 4 is unequal.

Thus with respect to even and odd numbers we find for basic 4D-Pythagorean quintuples the two possible cases

$$(a,b,c,d\,|\,e)=\left(2k+1,2l+1,2m+1,2n+1\,|\,2(2q+1)\right),$$
$$(a,b,c,d\,|\,e)=\left(2k+1,2l,2m,2n\,|\,2p+1\right), \qquad (5.8)$$

under which one has to find the cases which in addition satisfy the equality (5.1).

In contrast to basic 3D-Pythagorean quadruples where only the odd case $d=2p+1$ is possible for basic 4D-Pythagorean quintuples both the even case $e=2p$ and the odd case $e=2p+1$ are possible.

## 7. Basic 4D-Pythagorean Quintuples $(a,b,c,d\,|\,e)$ up to $e^2<500$

An arbitrary of the 24 permutation of the quadruple of numbers $(a,b,c,d)$ leads to equivalent quadruples. We make here an ordering of these quadruples with increasing $e$ and for equal $e$ we order the quadruples $(a,b,c,d)$ by $a\le b\le c\le d$.

By computer we found the following basic 4D-Pythagorean quintuples with even $e=2p=2(2q+1)$

$$\begin{aligned}&(1,1,1,1\,|\,2),\ (1,1,3,5\,|\,6),\ (1,1,7,7\,|\,10),\ (1,3,3,9\,|\,10),\\&(1,5,5,7\,|\,10),\ (1,1,5,13\,|\,14),\ (1,5,7,11\,|\,14),\ (3,3,3,13\,|\,14),\\&(3,5,9,9\,|\,14),\ (5,5,5,11\,|\,14),\ (1,3,5,17\,|\,18),\ (1,7,7,15\,|\,18),\\&(1,9,11,11\,|\,18),\ (5,5,7,15\,|\,18),\ (5,7,9,13\,|\,18),\ (1,1,11,19\,|\,22),\\&(1,5,13,17\,|\,22),\ (3,5,15,15\,|\,22),\ (3,9,13,15\,|\,22),\ (5,7,7,19\,|\,22),\\&(5,7,11,17\,|\,22),\ (5,11,13,13\,|\,22),\end{aligned} \qquad (6.1)$$

and with odd $e=2p+1$

$$\begin{aligned}
&(1,2,2,4\,|\,5), \quad (1,4,4,4\,|\,7), \quad (2,2,4,5\,|\,7), \quad (2,2,3,8\,|\,9),\\
&(2,4,5,6\,|\,9), \quad (1,2,4,10\,|\,11), \quad (2,2,7,8\,|\,11), \quad (4,4,5,8\,|\,11),\\
&(1,2,8,10\,|\,13), \quad (2,4,7,10\,|\,13), \quad (4,4,4,11\,|\,13), \quad (4,5,8,8\,|\,13),\\
&(4,6,6,9\,|\,13), \quad (1,4,8,12\,|\,15), \quad (2,3,4,14\,|\,15), \quad (2,4,6,13\,|\,15),\\
&(2,6,8,11\,|\,15), \quad (3,4,10,10,|15), \quad (4,4,7,12\,|\,15), \quad (4,8,8,9\,|\,15),\\
&(5,6,8,10\,|\,15), \quad (1,4,4,16\,|\,17), \quad (2,2,5,16\,|\,17), \quad (2,4,10,13\,|\,17),\\
&(2,5,8,14\,|\,17), \quad (2,8,10,11\,|\,17), \quad (3,6,10,12\,|\,17), \quad (5,8,10,10\,|\,17),\\
&(1,2,10,16\,|\,19), \quad (1,8,10,14\,|\,19), \quad (2,2,8,17\,|\,19), \quad (3,8,12,12\,|\,19),\\
&(4,7,10,14\,|\,19), \quad (6,6,8,15\,|\,19), \quad (6,9,10,12\,|\,19), \quad (8,8,8,13\,|\,19),\\
&(1,2,6,20\,|\,21), \quad (1,4,10,18\,|\,21), \quad (1,10,12,14\,|\,21), \quad (2,2,12,17\,|\,21),\\
&(2,4,14,15\,|\,21), \quad (2,7,8,18\,|\,21), \quad (2,9,10,16\,|\,21), \quad (3,4,4,20\,|\,21),\\
&(4,5,12,16\,|\,21), \quad (4,6,10,17\,|\,21), \quad (4,10,10,15\,|\,21), \quad (6,7,10,16\,|\,21),\\
&(8,8,12,13\,|\,21), \quad (8,9,10,14\,|\,21),
\end{aligned} \tag{6.2}$$

up to $e^2 < 500$, ($e \le 22$). This filtering by computer is not a method which provides only the basic 4D-Pythagorean quintuples with minimal expenditure.

## 8. Can the 4-Square Identity Provide a Parametrization of the 4D-Pythagorean Quintuples?

Relation (5.1) contains 5 integer parameters $(a,b,c,d,e)$ with 1 constraint and the 4-square identity in the specialized form (A.14) contains 4 independent parameters and on the right-hand side of last relation we have 4 squared numbers. Therefore one may pose the problem whether or not the parameters $(r_0,\boldsymbol{r})$ can provide a parametrization of the integers $(a,b,c,d,e)$. For this purpose we make the proposition

$$(a,b,c,d\,|\,e) = \left(r_0^2 - \boldsymbol{r}^2, 2r_0r_1, 2r_0r_2, 2r_0r_3\,|\,r_0^2 + \boldsymbol{r}^2\right), \tag{7.1}$$

From this follows

$$\begin{aligned}
&r_0^2 = \frac{e+a}{2}, \quad \boldsymbol{r}^2 = r_1^2 + r_2^2 + r_3^2 = \frac{e-a}{2}, \quad \Rightarrow\\
&2r_0 = \sqrt{2(e+a)}, \quad r_1 = \frac{b}{2r_0}, \quad r_2 = \frac{c}{2r_0}, \quad r_3 = \frac{d}{\sqrt{2r_0}},
\end{aligned} \tag{7.2}$$

or, equivalently

$$(r_0,r_1,r_2,r_3) = \frac{1}{\sqrt{2(e+a)}}(e+a,b,c,d), \tag{7.3}$$

and we may check again

$$\begin{aligned}
&\boldsymbol{r}^2 = r_1^2 + r_2^2 + r_3^2 = \frac{b^2+c^2+d^2}{2(e+a)} = \frac{e^2-a^2}{2(e+a)} = \frac{e-a}{2},\\
&r_0^2 + \boldsymbol{r}^2 = e, \quad r_0^2 - \boldsymbol{r}^2 = a.
\end{aligned} \tag{7.4}$$

plus possible permutations of $(a,b,c,d)$ where we used (5.1). Thus we have parameterized the integers $(a,b,c,d\,|\,e)$ by the parameters $(r_0,r_1,r_2,r_3)$ which become proportional to a quadruple of integers $(e+a,b,c,d)$ multiplied by a mostly common irrational number $\frac{1}{\sqrt{2(e+a)}}$. Furthermore, this parametrization is, in general, not unique since we can order the parameters $(a,b,c,d)$ by permutations in 4! = 24 ways (if all $(a,b,c,d)$ are different, in other cases fewer ones) and obtain then the same number of different parameterizations of a considered 4D-Pythagorean quintuple. This is a consequence that the relation (7.1) between the two kinds of parameterizations is not linear but quadratical. However, given a parametrization $(r_0,r_1,r_3,r_3)$ of the form (7.3) the reconstruction of $(a,b,c,d\,|\,e)$ is also not unique since an integer *f* can be split in different way into a sum $f=e+a$ of two different integers *e* and *a*.

We give four examples of the parametrization of $(a,b,c,d\,|\,e)$ by $(r_0,r_1,r_3,r_3)$, the first two from (6.1)

$$(1,1,1,1|2):\quad (r_0,r_1,r_2,r_3)=\frac{1}{\sqrt{6}}(3,1,1,1),$$

$$\begin{cases}(1,3,5,17|18):\quad (r_0,r_1,r_2,r_3)=\frac{1}{\sqrt{38}}(19,3,5,17)\\ (3,1,5,17|18):\quad (r_0,r_1,r_2,r_3)=\frac{1}{\sqrt{42}}(21,1,5,17)\\ (5,1,3,17|18):\quad (r_0,r_1,r_2,r_3)=\frac{1}{\sqrt{46}}(23,1,3,17)\\ (17,1,3,5|18):\quad (r_0,r_1,r_2,r_3)=\frac{1}{\sqrt{70}}(35,1,3,5),\end{cases}\tag{7.5}$$

and the second two from (6.2)

$$\begin{cases}(1,2,2,4|5):\quad (r_0,r_1,r_2,r_3)=\frac{1}{\sqrt{12}}(6,2,2,4)=\frac{1}{\sqrt{3}}(3,1,1,2),\\ (2,1,2,4|5):\quad (r_0,r_1,r_2,r_3)=\frac{1}{\sqrt{14}}(7,1,2,4),\\ (4,1,2,2|5)):\quad (r_0,r_1,r_2,r_3)=\frac{1}{\sqrt{18}}(9,1,2,2)=\frac{1}{3\sqrt{2}}(9,1,2,2),\end{cases}$$

$$\begin{cases}(2,4,5,6|9):\quad (r_0,r_1,r_2,r_3)=\frac{1}{\sqrt{22}}(11,4,5,6),\\ (4,2,5,6|9):\quad (r_0,r_1,r_2,r_3)=\frac{1}{\sqrt{26}}(13,2,5,6),\\ (5,2,4,6|9):\quad (r_0,r_1,r_2,r_3)=\frac{1}{\sqrt{28}}(14,2,4,6)=\frac{1}{\sqrt{7}}(7,1,2,3),\\ (6,2,4,5|9):\quad (r_0,r_1,r_2,r_3)=\frac{1}{\sqrt{30}}(15,2,4,5).\end{cases}\tag{7.6}$$

Although possible, such parametrizations by $(r_0,\boldsymbol{r})$ seem to be not favorable.

One may give also $(r_0,r_1,r_2,r_3)$ as integers and can then determine $(a,b,c,d\,|\,e)$ according to (7.1) and obtains integer values for $(a,b,c,d,e)$ which satisfy Equation (5.1). However, they are in most cases not basic quintuples as the following examples show

$$\begin{aligned}
&(r_0,r_1,r_2,r_3)=(2,1,1,1):\quad (1,4,4,4\,|\,7),\\
&(r_0,r_1,r_2,r_3)=(3,1,1,1):\quad (6,6,6,6\,|\,12)=6(1,1,1,1\,|\,2),\\
&(r_0,r_1,r_2,r_3)=(3,1,1,2):\quad (3,6,6,12\,|\,15)=3(1,2,2,4\,|\,5),\\
&(r_0,r_1,r_2,r_3)=(4,1,1,1):\quad (13,8,8,8\,|\,19),\\
&(r_0,r_1,r_2,r_3)=(4,1,1,2):\quad (10,8,8,16\,|\,22)=2(5,4,4,8\,|\,11),\\
&(r_0,r_1,r_2,r_3)=(4,1,2,2):\quad (7,8,16,16\,|\,25),\\
&(r_0,r_1,r_2,r_3)=(4,1,2,3):\quad (2,8,16,24\,|\,30)=2(1,4,8,12\,|\,15),\\
&(r_0,r_1,r_2,r_3)=(4,2,2,2):\quad (4,16,16,16\,|\,28)=4(1,4,4,4\,|\,7).
\end{aligned} \tag{7.7}$$

This procedure does not provide all basic quintuples $(a,b,c,d\,|\,e)$ in a systematically ordered way and, furthermore, it provides not only basic quintuples. In addition, this parametrization is only a modification of that in (7.1) and (7.2).

It is obvious from (A.16) that the considered results of this Section can be generalized to $N$D-Pythagorean $(N+1)$D-tuples. A parametrization of the 3D-Pythagorean quadruples is achieved if we set $r_3=0$ in formulae (7.1) - (7.3) but also with problem that it does not only provide basic quadruples.

## 9. Conclusions

We first investigated the three-dimensional generalization of Pythagorean numbers concerning the possibilities of integer solutions $(a,b,c\,|\,d)$ of the Diophantine equation $a^2+b^2+c^2=d^2$ with $d$ the spatial diagonal of a cuboid and called them 3D-Pythagorean quadruples. The basic solutions to the problem which means that the numbers $(a,b,c,d)$ should not possess a common divisor were given in ordered form up to $d^2<1000$ and a general parametrization of such numbers was derived. Then we extended the problem to the four-dimensional generalization of 3D-Pythagorean quadruples to 4D-Pythagorean quintuples that means basic integer solutions $(a,b,c,d\,|\,e)$ of the equation
$a^2+b^2+c^2+d^2=e^2$ which are the analogues in a four-dimensional Euclidean space and gave them up to $e^2<500$.

Every new result raises new problems and a few of them were formulated. As it is well known the Number Theory creates strange problems and conjectures from which some are very difficult to prove or to disprove. Many of them can be also ordered into the realm of Recreation Mathematics.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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## Appendix

Coordinate-invariant derivation of the 4-square identity from the algebra of real quaternions:

We derive in this Appendix the 4-square identity from the algebra of quaternions.

Quaternions $r$ form a closed algebra $\mathbb{H}$ of numbers discovered by W. R. Hamilton which are linear combinations of 4 basic numbers $1, i, j, k$ of the form (e.g., Stillwell [4], Conway and Smith [8], Gürlebeck and Sprössig [9], Koecher and Remmert [10] included in the collective book [11] and many others)

$$r = 1r_0 + ir_1 + jr_2 + kr_3 \equiv (r_0, \boldsymbol{r}), \quad \boldsymbol{r} = (r_1, r_2, r_3), \tag{A.1}$$

where the basic numbers satisfy the following multiplication rules

$$i^2 = j^2 = k^2 = -1, \quad jk = -kj = i, \quad ki = -ik = j, \quad ij = -ji = k, \tag{A.2}$$

from which follows

$$ijk = jki = kij = -ikj = -jik = -kji = -1. \tag{A.3}$$

To make the following derivations widely coordinate-invariant that maybe helps in a deeper understanding we use the representation (A.1) of a quaternion $r = (r_0, \boldsymbol{r})$ by a scalar part $r_0$ and a vectorial part $\boldsymbol{r}$ with 3 components.

The product of two quaternions $r = (r_0, \boldsymbol{r})$ and $s = (s_0, \boldsymbol{s})$ can be defined as follows

$$rs = (r_0, \boldsymbol{r})(s_0, \boldsymbol{s}) = (r_0 s_0 - \boldsymbol{rs}, s_0 \boldsymbol{r} + r_0 \boldsymbol{s} + [\boldsymbol{r}, \boldsymbol{s}]), \tag{A.4}$$

where

$$\boldsymbol{rs} = \boldsymbol{sr} = r_1 s_1 + r_2 s_2 + r_3 s_3, \tag{A.5}$$

means the symmetric scalar product and

$$[\boldsymbol{r}, \boldsymbol{s}] = -[\boldsymbol{s}, \boldsymbol{r}] = (r_2 s_3 - r_3 s_2, r_3 s_1 - r_1 s_3, r_1 s_2 - r_2 s_1), \tag{A.6}$$

the antisymmetric vector product of $\boldsymbol{r}$ and $\boldsymbol{s}$ in components with respect to an orthogonal basis. The product of quaternions is associative but, in general, noncommutative.

The quaternion

$$\overline{r} = \overline{(r_0, \boldsymbol{r})} \equiv (r_0, -\boldsymbol{r}), \tag{A.7}$$

is called the conjugate quaternion to $r = (r_0, \boldsymbol{r})$. Its product with $r$ is

$$\begin{aligned} r\overline{r} &= (r_0, \boldsymbol{r})(r_0, -\boldsymbol{r}) = (r_0^2 + \boldsymbol{r}^2, \boldsymbol{0}) = \overline{r}r, \quad \Rightarrow \\ r^{-1} &= \frac{\overline{r}}{r_0^2 + \boldsymbol{r}^2} = \frac{1}{r_0^2 + \boldsymbol{r}^2}(r_0, -\boldsymbol{r}), \quad r^{-1}r = rr^{-1} = (1, \boldsymbol{0}), \end{aligned} \tag{A.8}$$

and it is proportional to the identical quaternion $(1, \boldsymbol{0})$ and $r^{-1} \equiv \frac{\overline{r}}{r_0^2 + \boldsymbol{r}^2}$ is the reciprocal quaternion to $r$. The nonnegative number

$$|r| \equiv \sqrt{r_0^2 + \boldsymbol{r}^2}, \tag{A.9}$$

is the modulus or norm of a quaternion. For the conjugate quaternion $\overline{rs}$ to the product $rs$ one finds the relation

$$\overline{rs} = \left(r_0 s_0 - \boldsymbol{rs}, -s_0\boldsymbol{r} - r_0\boldsymbol{s} - [\boldsymbol{r},\boldsymbol{s}]\right) = \left(s_0, -\boldsymbol{s}\right)\left(r_0, -\boldsymbol{r}\right) = \overline{s}\,\overline{r}. \tag{A.10}$$

From this follows

$$\begin{aligned} rs\overline{s}\,\overline{r} &= \left(r\left(s\overline{s}\right)\overline{r}\right) = \left(r_0, \overline{r}\right)\left(s_0^2 + \boldsymbol{s}^2, \boldsymbol{0}\right) = \left(r_0^2 + \boldsymbol{r}^2, \boldsymbol{0}\right)\left(s_0^2 + \boldsymbol{s}^2, \boldsymbol{0}\right) \\ &= \left(\left(r_0^2 + \boldsymbol{r}^2\right)\left(s_0^2 + \boldsymbol{s}^2\right), \boldsymbol{0}\right), \\ rs\overline{s}\,\overline{r} &= \left(rs\right)\left(\overline{rs}\right) = \left(r_0 s_0 - \boldsymbol{rs}, s_0\boldsymbol{r} + r_0\boldsymbol{s} + [\boldsymbol{r},\boldsymbol{s}]\right)\left(r_0 s_0 - \boldsymbol{rs}, -s_0\boldsymbol{r} - r_0\boldsymbol{s} - [\boldsymbol{r},\boldsymbol{s}]\right) \\ &= \left(\left(r_0 s_0 - \boldsymbol{rs}\right)^2 + \left(s_0\boldsymbol{r} + r_0\boldsymbol{s} + [\boldsymbol{r},\boldsymbol{s}]\right)^2, \boldsymbol{0}\right). \end{aligned} \tag{A.11}$$

Thus by comparison of both expressions for $rs\overline{s}\,\overline{r}$ in (A.11) we find the (4-square) identity

$$\left(r_0^2 + \boldsymbol{r}^2\right)\left(s_0^2 + \boldsymbol{s}^2\right) = \left(r_0 s_0 - \boldsymbol{rs}\right)^2 + \underbrace{\left(s_0\boldsymbol{r} + r_0\boldsymbol{s} + [\boldsymbol{r},\boldsymbol{s}]\right)^2}_{\text{squared modulus of 3D-vector}}, \tag{A.12}$$

or the right-hand side written in components with respect to an orthogonal basis of the vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ ([4] [8])

$$\begin{aligned} \left(r_0 s_0 - \boldsymbol{rs}\right)^2 &= \left(r_0 s_0 - r_1 s_1 - r_2 s_2 - r_3 s_3\right)^2, \\ \left(s_0\boldsymbol{r} + r_0\boldsymbol{s} + [\boldsymbol{r},\boldsymbol{s}]\right)^2 &= +\left(s_0 r_1 + r_0 s_1 + r_2 s_3 - r_3 s_2\right)^2 \\ &\quad + \left(s_0 r_2 + r_0 s_2 + r_3 s_1 - r_1 s_3\right)^2 \\ &\quad + \left(s_0 r_3 + r_0 s_3 + r_1 s_2 - r_2 s_1\right)^2. \end{aligned} \tag{A.13}$$

The right-hand side of (A.12) is the sum of 4 squared integers but the left-hand side is only in special cases a squared integer. The right-hand side remains unchanged if one makes an even permutation of the 3 indices $(1,2,3)$. Many special cases of (A.13) can be considered.

In the first special case of $r = s$ or $\left(r_0 = s_0, \boldsymbol{r} = \boldsymbol{s}\right)$ from (A.12) follows the relation

$$\left(r_0^2 + \boldsymbol{r}^2\right)^2 = \left(r_0^2 - \boldsymbol{r}^2\right)^2 + \left(2r_0\boldsymbol{r}\right)^2, \tag{A.14}$$

or written in orthogonal components $\left(r_1, r_2, r_3\right)$ of the vector $\boldsymbol{r}$,

$$\begin{aligned} &\left(r_0^2 + \boldsymbol{r}^2\right)^2 = \left(r_0^2 + r_1^2 + r_2^2 + r_3^2\right)^2, \quad \left(r_0^2 - \boldsymbol{r}^2\right)^2 = \left(r_0^2 - r_1^2 - r_2^2 - r_3^2\right)^2, \\ &\left(2r_0\boldsymbol{r}\right)^2 = \left(2r_0 r_1\right)^2 + \left(2r_0 r_2\right)^2 + \left(2r_0 r_3\right)^2. \end{aligned} \tag{A.15}$$

The left-hand side of (A.14) is a squared integer and the right-hand side is the sum of 4 squared integers and this identity possesses 4 independent parameters.

Relation (A.14) can immediately be generalized to the $N$-dimensional case by substituting the 3D-vector $\boldsymbol{r}$ by an $N$D-vector $\boldsymbol{r} = \left(r_1, \cdots, r_N\right)$ where instead of (A.14) we have then the identity

$$\begin{aligned} &\left(r_0^2 + \boldsymbol{r}^2\right)^2 = \left(r_0^2 + \sum_{k=1}^{N} r_k^2\right)^2, \quad \left(r_0^2 - \boldsymbol{r}^2\right)^2 = \left(r_0^2 - \sum_{k=1}^{N} r_k^2\right)^2, \\ &\left(2r_0\boldsymbol{r}\right)^2 = \sum_{k=1}^{N}\left(2r_0 r_k\right)^2. \end{aligned} \tag{A.16}$$

It contains $N+1$ independent parameters $(r_0, r_1, \ldots, r_N)$.

In the second special case when $r_0^2 + \boldsymbol{r}^2$ or $s_0^2 + \boldsymbol{s}^2$ are squared integers they provide already separately cases for 4D-Pythagorean quintuples. If both are squared integers then (A.12) together with (A.13) provides a parametrization for 4D-Pythagorean quintuples, however, with 8 parameters. In these cases it can be true that for some of the numbers $(a,b,c,d)$ result negative integers which, however, squared become positive squares and satisfy relation (A.12). Such a case of $\left(r_0, r_1, r_2, r_3 \,|\, \sqrt{r_0^2 + \boldsymbol{r}^2}\right)$ and $\left(s_0, s_1, s_2, s_3 \,|\, \sqrt{s_0^2 + \boldsymbol{s}^2}\right)$ is $(1,1,1,1\,|\,2)$ and $(6,5,4,2\,|\,9)$ with resulting $(-5,9,13,7\,|\,18)$. One of the numbers $(a,b,c,d)$ can even become zero as it is the case, for example, for $(1,1,1,1\,|\,2)$ and $(5,3,1,1\,|\,6)$ from which results $(0,8,4,8\,|\,12)$ which is equivalent to the basic 3D-Pythagorean quadruple $(2,1,2\,|\,3)$.