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A Study on Generalized p-Oresme Numbers

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, we introduce the generalized p-Oresme sequences and we deal with, in detail, three special cases which we call them modified p-Oresme, p-Oresme-Lucas and p-Oresme sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

Keywords: Oresme numbers; Oresme-Lucas numbers; generalized Fibonacci numbers; modified p-Oresme numbers; p-Oresme-Lucas numbers; p-Oresme numbers.

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1 INTRODUCTION

p-Oresme numbers (see, for example, Cook [1]) are defined by the recurence relation

$$O_{n+2} = O_{n+1} - \frac{1}{p^2}O_n, \quad O_0 = 0, O_1 = \frac{1}{p}, \ \ p \neq 0.$$

The case $p = 2$, which is called the Oresme sequence, $\{O_n\}_{n\geq 0}$, was introduced by Nicole Oresme (1320–1382) in the 14-th century. Oresme obtained the sum of the rational numbers formed by the terms $0, \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \frac{6}{64}, ..., \frac{n}{2^n}$. These numbers form a second order sequence and are defined by the recurence relation

$$O_{n+2} = O_{n+1} - \frac{1}{4}O_n, \quad O_0 = 0, O_1 = \frac{1}{2}.$$

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In [2], Horadam presented a history and obtained an abundance of properties of these numbers. Oresme numbers have many interesting properties and applications in many fields of science (see, for example, [3],[4],[1],[2],[5],[6]).

The purpose of this article is to generalize and investigate these interesting sequence of numbers (i,e,, p-Oresme numbers). First, we recall some properties of Fibonacci numbers and its generalizations, namely generalized Fibonacci numbers.

The Fibonacci numbers and their generalizations have many interesting properties and applications to almost every field such as architecture, nature, art, physics and engineering. The sequence of Fibonacci numbers $\{F_n\}_{n\geq 0}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \ \ n \geq 2, \quad F_0 = 0, \ F_1 = 1.$$

The generalization of Fibonacci sequence leads to several nice and interesting sequences. The generalized Fibonacci sequence (or generalized (r,s)-sequence or Horadam sequence or 2-step Fibonacci sequence) $\{W_n(W_0,W_1;r,s)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined (by Horadam [7]) as follows:

$$W_n = rW_{n-1}+sW_{n-2}, \quad W_0 = a, W_1 = b, \ \ n \geq 2 \tag{1.1}$$

where W_0, W_1 are arbitrary complex (or real) numbers and r,s are real numbers, see also Horadam [8],[9],[10] and Soykan [11].

For some specific values of a,b,r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r,s and initial values.

Table 1. A few special case of generalized Fibonacci sequences

Name of sequence	$W_n(a,b;r,s)$	Binet Formula	OEIS[12]
Fibonacci	$W_n(0,1;1,1) = F_n$	$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$	A000045
Lucas	$W_n(2,1;1,1) = L_n$	$\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$	A000032
Pell	$W_n(0,1;2,1) = P_n$	$\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$	A000129
Pell-Lucas	$W_n(2,2;2,1) = Q_n$	$(1+\sqrt{2})^n + (1-\sqrt{2})^n$	A002203
Jacobsthal	$W_n(0,1;1,2) = J_n$	$\frac{2^n-(-1)^n}{3}$	A001045
Jacobsthal-Lucas	$W_n(2,1;1,2) = j_n$	$2^n + (-1)^n$	A014551

Here, OEIS stands for On-line Encyclopedia of Integer Sequences.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1,2,3,...$ when $s \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

Now we define two special cases of the sequence $\{W_n\}$. (r,s) sequence $\{G_n(0,1;r,s)\}_{n\geq 0}$ and Lucas (r,s) sequence $\{H_n(2,r;r,s)\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$G_{n+2} = rG_{n+1} + sG_n, \quad G_0 = 0, G_1 = 1, \tag{1.2}$$

$$H_{n+2} = rH_{n+1} + sH_n, \quad H_0 = 2, H_1 = r, \tag{1.3}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{r}{s}G_{-(n-1)} + \frac{1}{s}G_{-(n-2)}, \\ H_{-n} &= -\frac{r}{s}H_{-(n-1)} + \frac{1}{s}H_{-(n-2)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.2)-(1.3) hold for all integer n.

Some special cases of (r, s) sequence $\{G_n(0, 1; r, s)\}_{n\geq 0}$ and Lucas (r, s) sequence $\{H_n(2, r; r, s)\}_{n\geq 0}$ are as follows:

1. $G_n(0, 1; 1, 1) = F_n$, Fibonacci sequence,
2. $H_n(2, 1; 1, 1) = L_n$, Lucas sequence,
3. $G_n(0, 1; 2, 1) = P_n$, Pell sequence,
4. $H_n(2, 2; 2, 1) = Q_n$, Pell-Lucas sequence,
5. $G_n(0, 1; 1, 2) = J_n$, Jacobsthal sequence,
6. $H_n(2, 1; 1, 2) = j_n$, Jacobsthal-Lucas sequence.

The following theorem shows that the generalized Fibonacci sequence W_n at negative indices can be expressed by the sequence itself at positive indices.

Theorem 1.1. *For $n \in \mathbb{Z}$, for the generalized Fibonacci sequence (or generalized (r, s)-sequence or Horadam sequence or 2-step Fibonacci sequence) we have the following:*

(a)

$$\begin{aligned} W_{-n} &= (-1)^{-n-1}s^{-n}(W_n - H_nW_0) \\ &= (-1)^{n+1}s^{-n}(W_n - H_nW_0). \end{aligned}$$

(b)

$$W_{-n} = \frac{(-1)^{n+1}s^{-n}}{-W_1^2 + sW_0^2 + rW_0W_1}((2W_1 - rW_0)W_0W_{n+1} - (W_1^2 + sW_0^2)W_n).$$

Proof. For the proof, see Soykan [[13], Theorem 3.2 and Theorem 3.3]. □

The following theorem presents sum formulas of generalized (r, s) numbers (generalized Fibonacci numbers).

Theorem 1.2. *Let x be a real (or complex) number. For all integers m and j, for generalized (r, s) numbers (generalized Fibonacci numbers), we have the following sum formulas:*

(a) *If $(-s)^m x^2 - xH_m + 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{((-s)^m x - H_m)x^{n+1}W_{mn+j} + (-s)^m x^{n+1}W_{mn+j-m} + W_j - (-s)^m xW_{j-m}}{(-s)^m x^2 - xH_m + 1}. \tag{1.4}$$

(b) *If $(-s)^m x^2 - xH_m + 1 = u(x - a)(x - b) = 0$ for some $u, a, b \in \mathbb{C}$ with $u \neq 0$ and $a \neq b$, i.e., $x = a$ or $x = b$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{(x(n+2)(-s)^m - (n+1)H_m)x^n W_{j+mn} + (-s)^m (n+1)x^n W_{mn+j-m} - (-s)^m W_{j-m}}{2(-s)^m x - H_m}.$$

(c) *If $(-s)^m x^2 - xH_m + 1 = u(x - c)^2 = 0$ for some $u, c \in \mathbb{C}$ with $u \neq 0$, i.e., $x = c$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{(n+1)\left((-s)^m (n+2)x^n - nx^{n-1}H_m\right)W_{mn+j} + n(n+1)(-s)^m x^{n-1}W_{mn+j-m}}{2(-s)^m}.$$

Proof. It is given in Soykan [13, Theorem 4.1]. □

Note that (1.4) can be written in the following form

$$\sum_{k=1}^{n} x^k W_{mk+j} = \frac{((-s)^m x - H_m)x^{n+1} W_{mn+j} + (-s)^m x^{n+1} W_{mn+j-m} + x(H_m - (-s)^m x)W_j - (-s)^m x W_{j-m}}{(-s)^m x^2 - xH_m + 1}.$$

We give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence $\{W_n\}$.

Lemma 1.3. *Suppose that* $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ *is the ordinary generating function of the generalized Fibonacci sequence* $\{W_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} W_n x^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x}{1 - rx - sx^2}. \tag{1.5}$$

Proof. For a proof, see [[11], Lemma 1.1]. □

1.1 Binet's Formula for the Distinct Roots Case and Single Root Case

Let α and β be two roots of the quadratic equation

$$x^2 - rx - s = 0, \tag{1.6}$$

of which the left-hand side is called the characteristic polynomial (or the characteristic equation) of the recurrence relation (1.1). The following theorem presents the Binet's formula of the sequence W_n.

Theorem 1.4. *The general term of the sequence* W_n *can be presented by the following Binet's formula:*

$$\begin{aligned} W_n &= \begin{cases} \frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n & , \text{ if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ (nW_1 - \alpha(n-1)W_0)\alpha^{n-1} & , \text{ if } \alpha = \beta \text{ (Single Root Case)} \end{cases} \\ &= \begin{cases} \frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n & , \text{ if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ (nW_1 - \frac{r}{2}(n-1)W_0)\left(\frac{r}{2}\right)^{n-1} & , \text{ if } \alpha = \beta \text{ (Single Root Case)} \end{cases}. \end{aligned}$$

Proof. For a proof, see Soykan [11] and [13]. □

The roots of characteristic equation are

$$\alpha = \frac{r + \sqrt{\Delta}}{2}, \ \ \beta = \frac{r - \sqrt{\Delta}}{2}. \tag{1.7}$$

where

$$\Delta = r^2 + 4s$$

and the followings hold

$$\begin{aligned} \alpha + \beta &= r, \\ \alpha\beta &= -s, \\ (\alpha - \beta)^2 &= (\alpha + \beta)^2 - 4\alpha\beta = r^2 + 4s. \end{aligned}$$

If $\Delta = r^2 + 4s \neq 0$ then $\alpha \neq \beta$ i.e., there are distinct roots of the quadratic equation (1.6) and if $\Delta = r^2 + 4s = 0$ then $\alpha = \beta$, i.e., there is a single root of the quadratic equation (1.6).

In the case $r^2 + 4s \neq 0$ so that $\alpha \neq \beta$, for all integers n, (r, s) and Lucas (r, s) numbers (using initial conditions in Theorem 1.4) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= \frac{\alpha^n}{(\alpha-\beta)} + \frac{\beta^n}{(\beta-\alpha)}, \\ H_n &= \alpha^n + \beta^n, \end{aligned}$$

respectively. In the case $r^2 + 4s = 0$ so that $\alpha = \beta$, for all integers n, (r, s) and Lucas (r, s) numbers (using initial conditions in Theorem 1.4) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= n\alpha^{n-1}, \\ H_n &= 2\alpha^n, \end{aligned}$$

respectively.

2 GENERALIZED P-ORESME SEQUENCE

From now, throughout the paper we assume that $0 \neq p \in \mathbb{R}$ unless otherwise stated. In this paper we consider the case $r = 1, s = -\frac{1}{p^2}$. For $0 \neq p \in \mathbb{R}$, a generalized p-Oresme sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relations

$$W_n = W_{n-1} - \frac{1}{p^2} W_{n-2} \tag{2.1}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = p^2 W_{-(n-1)} - p^2 W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) holds for all integer n. The case $p^2 = 4$ is given in Soykan [6].

Using standard techniques for solving recurrence relations, the auxiliary equation, and its roots are given by

$$\begin{aligned} x^2 - x + \frac{1}{p^2} &= 0 \\ &\Rightarrow \\ p^2x^2 - p^2x + 1 &= 0 \end{aligned} \tag{2.2}$$

and

$$\alpha = \frac{1}{2p}\left(p + \sqrt{p^2-4}\right), \ \beta = \frac{1}{2p}\left(p - \sqrt{p^2-4}\right).$$

Note that

$$\begin{aligned} \alpha + \beta &= 1. \\ \alpha\beta &= \frac{1}{p^2}, \\ \alpha - \beta &= \frac{\sqrt{p^2-4}}{p}. \end{aligned}$$

Note that if $p^2-4\neq 0$, i,e., $p^2\neq 4$, then $\alpha\neq\beta$ i.e., there are distinct roots of the quadratic equation (2.2) and if $p^2=4$ then $\alpha=\beta=\frac{1}{2}$, i.e., there is a single root of the quadratic equation (2.2). Therefore, by Theorem 1.4, the Binet formula can be written as

$$W_n = \begin{cases} \dfrac{W_1-\beta W_0}{\alpha-\beta}\alpha^n - \dfrac{W_1-\alpha W_0}{\alpha-\beta}\beta^n & , \text{ if } p^2\neq 4, \text{ i.e., } p\neq -2 \text{ and } p\neq 2 \\ \left(nW_1-\frac{1}{2}(n-1)W_0\right)\left(\frac{1}{2}\right)^{n-1} & , \text{ if } p^2=4, \text{ i.e., } p=-2 \text{ or } p=2 \end{cases} \quad (2.3)$$

$$= \begin{cases} \dfrac{\Psi}{2^{n+1}p^n\sqrt{p^2-4}} & , \text{ if } p^2\neq 4 \\ \left(nW_1-\frac{1}{2}(n-1)W_0\right)\left(\frac{1}{2}\right)^{n-1} & , \text{ if } p^2=4 \end{cases}$$

where

$$\Psi=(2pW_1-(p-\sqrt{p^2-4})W_0)(p+\sqrt{p^2-4})^n-(2pW_1-(p+\sqrt{p^2-4})W_0)(p-\sqrt{p^2-4})^n.$$

The first few generalized p-Oresme numbers with positive subscript and negative subscript are given in the following Table 2.

Table 2. A few generalized p-Oresme numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$-p^2W_1+p^2W_0$
2	$W_1-\frac{1}{p^2}W_0$	$-p^4W_1+p^2(p^2-1)W_0$
3	$\frac{p^2-1}{p^2}W_1-\frac{1}{p^2}W_0$	$-p^4(p^2-1)W_1+p^4(p^2-2)W_0$
4	$\frac{p^2-2}{p^2}W_1-\frac{p^2-1}{p^4}W_0$	$-p^6(p^2-2)W_1+p^4(p^4-3p^2+1)W_0$
5	$\frac{p^4-3p^2+1}{p^4}W_1-\frac{p^2-2}{p^4}W_0$	$-p^6(p^4-3p^2+1)W_1+p^6(p^4-4p^2+3)W_0$
6	$\frac{p^4-4p^2+3}{p^4}W_1-\frac{p^4-3p^2+1}{p^6}W_0$	$-p^8(p^4-4p^2+3)W_1+p^6(p^6-5p^4+6p^2-1)W_0$

Now we define three special cases of the sequence $\{W_n\}$. Modified p-Oresme sequence $\{G_n\}_{n\geq 0}$, p-Oresme-Lucas sequence $\{H_n\}_{n\geq 0}$ and p-Oresme sequence $\{O_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$G_{n+2} = G_{n+1}-\frac{1}{p^2}G_n, \quad G_0=0, G_1=1, \quad (2.4)$$

$$H_{n+2} = H_{n+1}-\frac{1}{p^2}H_n, \quad H_0=2, H_1=1, \quad (2.5)$$

$$O_{n+2} = O_{n+1}-\frac{1}{p^2}O_n, \quad O_0=0, O_1=\frac{1}{p}. \quad (2.6)$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{O_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= p^2G_{-(n-1)}-p^2G_{-(n-2)}, \\ H_{-n} &= p^2H_{-(n-1)}-p^2H_{-(n-2)}, \\ O_{-n} &= p^2O_{-(n-1)}-p^2O_{-(n-2)}, \end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (2.4)-(2.6) hold for all integer n.

Note that the case $p^2=4$ is investigated in Soykan [6]. Note also that 2-Oresme sequence is the Oresme sequence. Next, we present the first few values of the modified p-Oresme, p-Oresme-Lucas and p-Oresme numbers with positive and negative subscripts:

Table 3. The first few values of the special second-order numbers with positive and negative subscripts

n	0	1	2	3	4	5	6
G_n	0	1	1	$\frac{p^2-1}{p^2}$	$\frac{p^2-2}{p^2}$	$\frac{p^4-3p^2+1}{p^4}$	$\frac{p^4-4p^2+3}{p^4}$
G_{-n}		$-p^2$	$-p^4$	$-p^4(p^2-1)$	$-p^6(p^2-2)$	$-p^6(p^4-3p^2+1)$	$-p^8(p^4-4p^2+3)$
H_n	2	1	$\frac{p^2-2}{p^2}$	$\frac{p^2-3}{p^2}$	$\frac{p^4-4p^2+2}{p^4}$	$\frac{p^4-5p^2+5}{p^4}$	$\frac{p^6-6p^4+9p^2-2}{p^6}$
H_{-n}		p^2	p^4-2p^2	p^6-3p^4	$p^8-4p^6+2p^4$	$p^{10}-5p^8+5p^6$	$p^{12}-6p^{10}+9p^8-2p^6$
O_n	0	$\frac{1}{p}$	$\frac{1}{p}$	$\frac{p^2-1}{p^3}$	$\frac{p^2-2}{p^3}$	$\frac{p^4-3p^2+1}{p^5}$	$\frac{p^4-4p^2+3}{p^5}$
O_{-n}		$-p$	$-p^3$	$-p^3(p^2-1)$	$-p^5(p^2-2)$	$-p^5(p^4-3p^2+1)$	$-p^7(p^4-4p^2+3)$

For all integers n, modified p-Oresme, p-Oresme-Lucas and p-Oresme numbers (using initial conditions in (2.3)) can be expressed using Binet's formulas as

$$G_n = \begin{cases} \dfrac{1}{2^n p^{n-1}\sqrt{p^2-4}}\left(\left(p+\sqrt{p^2-4}\right)^n - \left(p-\sqrt{p^2-4}\right)^n\right) & , \text{ if } p^2 \neq 4 \\ \dfrac{n}{2^{n-1}} & , \text{ if } p^2 = 4 \end{cases}$$

and

$$\begin{aligned} H_n &= \begin{cases} \dfrac{1}{2^n p^n}\left(\left(p+\sqrt{p^2-4}\right)^n + \left(p-\sqrt{p^2-4}\right)^n\right) & , \text{ if } p^2 \neq 4 \\ \dfrac{1}{2^{n-1}} & , \text{ if } p^2 = 4 \end{cases} \\ &= \dfrac{1}{2^n p^n}\left(\left(p+\sqrt{p^2-4}\right)^n + \left(p-\sqrt{p^2-4}\right)^n\right) \end{aligned}$$

and

$$O_n = \begin{cases} \dfrac{1}{2^n p^n\sqrt{p^2-4}}\left(\left(p+\sqrt{p^2-4}\right)^n - \left(p-\sqrt{p^2-4}\right)^n\right) & , \text{ if } p^2 \neq 4 \\ \dfrac{n}{2^{n-1}}\frac{1}{p} & , \text{ if } p^2 = 4 \end{cases}$$

respectively.

Note that

$$G_n = pO_n. \tag{2.7}$$

and

$$H_n = \begin{cases} \dfrac{1}{2^n p^n}\left(\left(p+\sqrt{p^2-4}\right)^n + \left(p-\sqrt{p^2-4}\right)^n\right) & , \text{ if } p^2 \neq 4 \\ \frac{1}{n}G_n & , \text{ if } p^2 = 4 \end{cases}$$

and

$$H_n = \begin{cases} \dfrac{1}{2^n p^n}\left(\left(p+\sqrt{p^2-4}\right)^n + \left(p-\sqrt{p^2-4}\right)^n\right) & , \text{ if } p^2 \neq 4 \\ \frac{p}{n}O_n & , \text{ if } p^2 = 4 \end{cases}.$$

From the last two equalities, we see that

$$G_n = \left\{ \; nH_n \quad , \quad \text{if } p^2 = 4 \right. \tag{2.8}$$

and

$$O_n = \left\{ \; \tfrac{n}{p}H_n \quad , \quad \text{if } p^2 = 4 \right. .$$

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} W_n x^n$ of the sequence $\{W_n\}$.

Lemma 2.1. *Suppose that* $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ *is the ordinary generating function of the generalized p-Oresme sequence* $\{W_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} W_n x^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n x^n = p^2 \times \frac{W_0 + (W_1 - W_0)x}{x^2 - p^2 x + p^2}.$$

Proof. In Lemma 1.3, take $r = 1, s = -\frac{1}{p^2}$. □
The previous Lemma gives the following results as particular examples.

Corollary 2.2. *Generated functions of modified p-Oresme, p-Oresme-Lucas and p-Oresme numbers are*

$$\begin{aligned}\sum_{n=0}^{\infty} G_n x^n &= \frac{p^2 x}{x^2 - p^2 x + p^2},\\ \sum_{n=0}^{\infty} H_n x^n &= \frac{p^2(2-x)}{x^2 - p^2 x + p^2},\\ \sum_{n=0}^{\infty} O_n x^n &= \frac{px}{x^2 - p^2 x + p^2},\end{aligned}$$

respectively.

Proof. In Lemma 2.1, take $W_n = G_n$ with $G_0 = 0, G_1 = 1$, $W_n = H_n$ with $H_0 = 2, H_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{p}$, respectively. □

3 SIMSON FORMULAS

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized p-Oresme sequence $\{W_n\}_{n\geq 0}$.

Theorem 3.1 (Simson Formula of Generalized p-Oresme Numbers)**.** *For all integers* n, *we have*

$$\begin{vmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{vmatrix} = -\frac{1}{p^{2n}}(p^2 W_1^2 + W_0^2 - p^2 W_0 W_1). \quad (3.1)$$

Proof. For a proof of Eq. (3.1), see Soykan [14], just take $s = -\frac{1}{p^2}$. □
The previous theorem gives the following results as particular examples.

Corollary 3.2. *For all integers* n, *modified p-Oresme, p-Oresme-Lucas and p-Oresme numbers have the following properties:*

$$\begin{aligned}\begin{vmatrix} G_{n+1} & G_n \\ G_n & G_{n-1} \end{vmatrix} &= -\frac{1}{p^{2n-2}}\\ \begin{vmatrix} H_{n+1} & H_n \\ H_n & H_{n-1} \end{vmatrix} &= \frac{p^2 - 4}{p^{2n}}\\ \begin{vmatrix} O_{n+1} & O_n \\ O_n & O_{n-1} \end{vmatrix} &= \frac{-1}{p^{2n}}\end{aligned}$$

respectively.

4 SOME IDENTITIES

In this section, we obtain some identities of generalized Oresme, modified Oresme, Oresme-Lucas and Oresme numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{G_n\}$.

Lemma 4.1. *The following equalities are true:*

$$\begin{aligned}
W_n &= p^4(-(p^2-1)W_1+(p^2-2)W_0)G_{n+4}-p^2(-p^2(p^2-2)W_1+(p^4-3p^2+1)W_0)G_{n+3}, \quad (4.1)\\
W_n &= p^2(-p^2W_1+(p^2-1)W_0)G_{n+3}+p^2((p^2-1)W_1+(2-p^2)W_0)G_{n+2},\\
W_n &= p^2(W_0-W_1)G_{n+2}+(p^2W_1+(1-p^2)W_0)G_{n+1},\\
W_n &= W_0G_{n+1}+(W_1-W_0)G_n,\\
W_n &= W_1G_n-\frac{W_0}{p^2}G_{n-1},
\end{aligned}$$

and

$$\begin{aligned}
(-p^2W_1^2-W_0^2+p^2W_0W_1)G_n &= p^6((p^2-1)W_1-W_0)W_{n+4}-p^4(p^2(p^2-2)W_1+(1-p^2)W_0)W_{n+3},\\
(-p^2W_1^2-W_0^2+p^2W_0W_1)G_n &= p^4(p^2W_1-W_0)W_{n+3}+p^4((1-p^2)W_1+W_0)W_{n+2},\\
(-p^2W_1^2-W_0^2+p^2W_0W_1)G_n &= p^4W_1W_{n+2}+p^2(W_0-p^2W_1)W_{n+1},\\
(-p^2W_1^2-W_0^2+p^2W_0W_1)G_n &= p^2W_0W_{n+1}-p^2W_1W_n,\\
(-p^2W_1^2-W_0^2+p^2W_0W_1)G_n &= p^2(W_0-W_1)W_n-W_0W_{n-1}.
\end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (4.1). To show (4.1), writing

$$W_n = a\times G_{n+4}+b\times G_{n+3}$$

and solving the system of equations

$$\begin{aligned}
W_0 &= a\times G_4+b\times G_3\\
W_1 &= a\times G_5+b\times G_4
\end{aligned}$$

we find that $a=p^4(-(p^2-1)W_1+(p^2-2)W_0), b=-p^2(-p^2(p^2-2)W_1+(p^4-3p^2+1)W_0)$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{H_n\}$ and $\{W_n\}$.

Lemma 4.2. *The following equalities are true:*

$$\begin{aligned}
(p^2-4)W_n &= p^4(p^2(p^2-3)W_1-(p^4-4p^2+2)W_0)H_{n+4}+p^4(-(p^4-4p^2+2)W_1+(p^4-5p^2+5)W_0)H_{n+3}\\
(p^2-4)W_n &= p^4((p^2-2)W_1+(3-p^2)W_0)H_{n+3}+p^2(-p^2(p^2-3)W_1+(p^4-4p^2+2)W_0)H_{n+2}\\
(p^2-4)W_n &= p^2(p^2W_1+(2-p^2)W_0)H_{n+2}+p^2((2-p^2)W_1+(p^2-3)W_0)H_{n+1}\\
(p^2-4)W_n &= p^2(2W_1-W_0)H_{n+1}+(-p^2W_1+(p^2-2)W_0)H_n\\
(p^2-4)W_n &= (p^2W_1-2W_0)H_n+(W_0-2W_1)H_{n-1}
\end{aligned}$$

and

$$\begin{aligned}
(p^2W_1^2+W_0^2-p^2W_0W_1)H_n &= p^4(p^2(p^2-3)W_1+(2-p^2)W_0)W_{n+4}-p^4((p^4-4p^2+2)W_1+(3-p^2)W_0)W_{n+3}, \\
(p^2W_1^2+W_0^2-p^2W_0W_1)H_n &= -p^4((2-p^2)W_1+W_0)W_{n+3}-p^2(p^2(p^2-3)W_1+(2-p^2)W_0)W_{n+2}, \\
(p^2W_1^2+W_0^2-p^2W_0W_1)H_n &= p^2(p^2W_1-2W_0)W_{n+2}-p^2((p^2-2)W_1-W_0)W_{n+1}, \\
(p^2W_1^2+W_0^2-p^2W_0W_1)H_n &= p^2(2W_1-W_0)W_{n+1}-(p^2W_1-2W_0)W_n, \\
(p^2W_1^2+W_0^2-p^2W_0W_1)H_n &= (p^2W_1-(p^2-2)W_0)W_n+(W_0-2W_1)W_{n-1}.
\end{aligned}$$

Now, we give a few basic relations between $\{W_n\}$ and $\{O_n\}$.

Lemma 4.3. *The following equalities are true:*

$$\begin{aligned}
W_n &= p^5((1-p^2)W_1+(p^2-2)W_0)O_{n+4}+p^3(p^2(p^2-2)W_1-(p^4-3p^2+1)W_0)O_{n+3}, \\
W_n &= p^3(-p^2W_1+(p^2-1)W_0)O_{n+3}+p^3((p^2-1)W_1+(2-p^2)W_0)O_{n+2}, \\
W_n &= p^3(W_0-W_1)O_{n+2}+p(p^2W_1+(1-p^2)W_0)O_{n+1}, \\
W_n &= pW_0O_{n+1}+p(W_1-W_0)O_n, \\
W_n &= pW_1O_n-\frac{W_0}{p}O_{n-1},
\end{aligned}$$

and

$$\begin{aligned}
(p^2W_1^2+W_0^2-p^2W_0W_1)O_n &= -p^5((p^2-1)W_1-W_0)W_{n+4}+p^3(p^2(p^2-2)W_1+(1-p^2)W_0)W_{n+3}, \\
(p^2W_1^2+W_0^2-p^2W_0W_1)O_n &= -p^3(p^2W_1-W_0)W_{n+3}-p^3((1-p^2)W_1+W_0)W_{n+2}, \\
(p^2W_1^2+W_0^2-p^2W_0W_1)O_n &= -p^3W_1W_{n+2}+p(p^2W_1-W_0)W_{n+1}, \\
(p^2W_1^2+W_0^2-p^2W_0W_1)O_n &= -pW_0W_{n+1}+pW_1W_n, \\
(p^2W_1^2+W_0^2-p^2W_0W_1)O_n &= p(W_1-W_0)W_n+\frac{1}{p}W_0W_{n-1}.
\end{aligned}$$

Next, we present a few basic relations between $\{G_n\}$ and $\{H_n\}$.

Lemma 4.4. *The following equalities are true:*

$$\begin{aligned}
p^2H_n &= p^6(p^2-3)G_{n+4}-p^4(p^4-4p^2+2)G_{n+3}, \\
p^2H_n &= p^4(p^2-2)G_{n+3}-p^4(p^2-3)G_{n+2}, \\
p^2H_n &= p^4G_{n+2}-p^2(p^2-2)G_{n+1}, \\
p^2H_n &= 2p^2G_{n+1}-p^2G_n, \\
p^2H_n &= p^2G_n-2G_{n-1},
\end{aligned}$$

and

$$\begin{aligned}
(p^2-4)G_n &= p^6(p^2-3)H_{n+4}-p^4(p^4-4p^2+2)H_{n+3}, \\
(p^2-4)G_n &= p^4(p^2-2)H_{n+3}-p^4(p^2-3)H_{n+2}, \\
(p^2-4)G_n &= p^4H_{n+2}+p^2(2-p^2)H_{n+1}, \\
(p^2-4)G_n &= 2p^2H_{n+1}-p^2H_n, \\
(p^2-4)G_n &= p^2H_n-2H_{n-1}.
\end{aligned}$$

Now, we give a few basic relations between $\{G_n\}$ and $\{O_n\}$.

Lemma 4.5. *The following equalities are true:*

$$\begin{aligned} p^2O_n &= -p^5(p^2-1)G_{n+4}+p^5(p^2-2)G_{n+3}, \\ p^2O_n &= -p^5G_{n+3}+p^3(p^2-1)G_{n+2}, \\ p^2O_n &= -p^3G_{n+2}+p^3G_{n+1}, \\ pO_n &= G_n, \end{aligned}$$

and

$$\begin{aligned} G_n &= -p^5(p^2-1)O_{n+4}+p^5(p^2-2)O_{n+3}, \\ G_n &= -p^5O_{n+3}+p^3(p^2-1)O_{n+2}, \\ G_n &= -p^3O_{n+2}+p^3O_{n+1}, \\ G_n &= pO_n. \end{aligned}$$

Next, we present a few basic relations between $\{H_n\}$ and $\{O_n\}$.

Lemma 4.6. *The following equalities are true:*

$$\begin{aligned} H_n &= p^5(p^2-3)O_{n+4}-p^3(p^4-4p^2+2)O_{n+3} \\ H_n &= p^3(p^2-2)O_{n+3}-p^3(p^2-3)O_{n+2} \\ H_n &= p^3O_{n+2}-p(p^2-2)O_{n+1} \\ H_n &= 2pO_{n+1}-pO_n \\ H_n &= pO_n-\frac{2}{p}O_{n-1} \end{aligned}$$

and

$$\begin{aligned} (p^2-4)O_n &= p^5(p^2-3)H_{n+4}-p^3(p^4-4p^2+2)H_{n+3}, \\ (p^2-4)O_n &= p^3(p^2-2)H_{n+3}-p^3\left(p^2-3\right)H_{n+2}, \\ (p^2-4)O_n &= p^3H_{n+2}-p(p^2-2)H_{n+1}, \\ (p^2-4)O_n &= 2pH_{n+1}-pH_n, \\ (p^2-4)O_n &= pH_n-\frac{2}{p}H_{n-1}. \end{aligned}$$

We now present a few special identities for the generalized p-Oresme sequence $\{W_n\}$.

Theorem 4.7. *(Catalan's identity of the generalized p-Oresme sequence) For all integers n and m, the following identity holds:*

$$W_{n+m}W_{n-m}-W_n^2=\begin{cases} \dfrac{(\alpha\beta)^{n-m}\left(\alpha^m-\beta^m\right)^2}{(\alpha-\beta)^2}\left(W_1-\beta W_0\right)\left(-W_1+\alpha W_0\right) & , \text{ if } p^2\neq 4 \\ -m^2\left(\frac{1}{2}\right)^{2n}\left(4W_1^2+W_0^2-4W_0W_1\right) & , \text{ if } p^2=4 \end{cases}.$$

Proof. Use the identity (2.3). □

As special cases of the above theorem, we have the following corollary.

Corollary 4.8. *For all integers n and m, the following identities hold:*

(a)

$$G_{n+m}G_{n-m}-G_n^2=\begin{cases} -\dfrac{1}{2^{2m}p^{2n-2}(p^2-4)}\left(\left(p+\sqrt{p^2-4}\right)^m-\left(p-\sqrt{p^2-4}\right)^m\right)^2 & , \text{ if } p^2\neq 4 \\ -4m^2\left(\frac{1}{2}\right)^{2n} & , \text{ if } p^2=4 \end{cases}.$$

(b)

$$H_{n+m}H_{n-m}-H_n^2=\begin{cases} \frac{1}{2^{2m}p^{2n}}\left(\left(p+\sqrt{p^2-4}\right)^m-\left(p-\sqrt{p^2-4}\right)^m\right)^2 & , \text{ if } p^2\neq 4 \\ 0 & , \text{ if } p^2=4 \end{cases}.$$

(c)

$$O_{n+m}O_{n-m}-O_n^2=\begin{cases} -\frac{1}{2^{2m}p^{2n}(p^2-4)}\left(\left(p+\sqrt{p^2-4}\right)^m-\left(p-\sqrt{p^2-4}\right)^m\right)^2 & , \text{ if } p^2\neq 4 \\ -m^2\left(\frac{1}{2}\right)^{2n}\frac{4}{p^2} & , \text{ if } p^2=4 \end{cases}.$$

Note that for $m=1$ in Catalan's identity of the generalized Oresme sequence, we get the Cassini (Simson) identity for the generalized p-Oresme sequnce.

Theorem 4.9. *(Cassini's identity of the generalized p-Oresme sequence) For all integers n, the following identity holds:*

$$W_{n+1}W_{n-1}-W_n^2=-\frac{1}{p^{2n}}(p^2W_1^2+W_0^2-p^2W_0W_1).$$

As special cases of the above theorem, we have the following corollary.

Corollary 4.10. *For all integers n, the following identities hold:*

(a) $G_{n+1}G_{n-1}-G_n^2=-\frac{1}{p^{2n-2}}.$

(b) $H_{n+1}H_{n-1}-H_n^2=\frac{p^2-4}{p^{2n}}.$

(c) $O_{n+1}O_{n-1}-O_n^2=\frac{-1}{p^{2n}}.$

The d'Ocagne's and Gelin-Cesàro's identities can also be obtained by using the identity (2.3). The next theorem presents d'Ocagne's and Gelin-Cesàro's identities of generalized p-Oresme sequence $\{W_n\}$.

Theorem 4.11. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$W_{m+1}W_n-W_mW_{n+1}=\begin{cases} -\frac{1}{\alpha-\beta}\left(W_1-\beta W_0\right)\left(W_1-\alpha W_0\right)\left(\alpha^m\beta^n-\alpha^n\beta^m\right) & , \text{ if } p^2\neq 4 \\ -2^{-m-n-1}\left(m-n\right)\left(W_0-2W_1\right)^2 & , \text{ if } p^2=4 \end{cases}$$

(b) *(Gelin-Cesàro's identity)*

$$W_{n+2}W_{n+1}W_{n-1}W_{n-2}-W_n^4=\begin{cases} \frac{(\alpha\beta)^{n-3}}{(\alpha-\beta)^2}\left(W_1-\alpha W_0\right)\left(W_1-\beta W_0\right)\Lambda_1 & , \text{ if } p^2\neq 4 \\ -\left(\frac{1}{2}\right)^{4n}\Lambda_2 & , \text{ if } p^2=4 \end{cases}$$

where

$\Lambda_1=(-(\alpha^2+\beta^2+3\alpha\beta)(\alpha^{2n}+\beta^{2n})\beta\alpha+(\alpha^4+\beta^4+4\alpha^2\beta^2+2\alpha\beta^3+2\alpha^3\beta)\beta^n\alpha^n)W_1^2+\alpha\beta(-(\alpha^2+\beta^2+3\alpha\beta)(\alpha^2\beta^{2n}+\alpha^{2n}\beta^2)+\alpha^n\beta^n(\alpha^4+\beta^4+4\alpha^2\beta^2+2\alpha\beta^3+2\alpha^3\beta))W_0^2+(2\alpha\beta(\alpha^2+\beta^2+3\alpha\beta)(\alpha\beta^{2n}+\alpha^{2n}\beta)-\alpha^n\beta^n(\alpha+\beta)(\alpha^4+\beta^4+4\alpha^2\beta^2+2\alpha\beta^3+2\alpha^3\beta))W_0W_1$

and

$\Lambda_2=16(5n^2-4)W_1^4+(5n^2-10n+1)W_0^4-16\left((10n^2-5n-8)\right)W_0W_1^3-4\left((10n^2-15n-3)\right)W_0^3W_1+4(30n^2-30n-19)W_0^2W_1^2.$

Proof. Use the identity (2.3). □

As special cases of the above theorem, we have the following three corollaries. First one presents d'Ocagne's and Gelin-Cesàro's identities of modified p-Oresme sequence $\{G_n\}$.

Corollary 4.12. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$G_{m+1}G_n - G_mG_{n+1} = \begin{cases} -\frac{p}{\sqrt{p^2-4}}\left(\alpha^m\beta^n - \alpha^n\beta^m\right) & , \text{ if } p^2 \neq 4 \\ -2^{-m-n+1}\left(m-n\right) & , \text{ if } p^2 = 4 \end{cases}.$$

(b) *(Gelin-Cesàro's identity)*

$$G_{n+2}G_{n+1}G_{n-1}G_{n-2} - G_n^4 = \begin{cases} \frac{1}{p^{2n-4}(p^2-4)}((p^4+4)\frac{1}{p^{2n}} - \left(p^2+1\right)\left(\alpha^n+\beta^n\right)^2) & , \text{ if } p^2 \neq 4 \\ -2^{-4n+4}\left(5n^2-4\right) & , \text{ if } p^2 = 4 \end{cases}$$

Second one presents d'Ocagne's and Gelin-Cesàro's identities of p-Oresme-Lucas sequence $\{H_n\}$.

Corollary 4.13. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$H_{m+1}H_n - H_mH_{n+1} = \begin{cases} \frac{\sqrt{p^2-4}}{p}\left(\alpha^m\beta^n - \alpha^n\beta^m\right) & , \text{ if } p^2 \neq 4 \\ 0 & , \text{ if } p^2 = 4 \end{cases}.$$

(b) *(Gelin-Cesàro's identity)*

$$H_{n+2}H_{n+1}H_{n-1}H_{n-2} - H_n^4 = \begin{cases} -\frac{1}{p^{2n-2}}\Omega & , \text{ if } p^2 \neq 4 \\ 0 & , \text{ if } p^2 = 4 \end{cases}$$

where
$\Omega = ((-\left(p^2+1\right)\left(\alpha^{2n}+\beta^{2n}\right)+\left(p^4-2p^2+2\right)p^{-2n})+4(-\left(p^2+1\right)\left(\alpha^2\beta^{2n}+\alpha^{2n}\beta^2\right)+p^{-2n-2}(p^4-2p^2+2))+2(2(p^2+1)(\alpha\beta^{2n}+\alpha^{2n}\beta)-p^{-2n}(p^4-2p^2+2)))$.

Third one presents d'Ocagne's and Gelin-Cesàro's identities of p-Oresme sequence $\{O_n\}$.

Corollary 4.14. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$O_{m+1}O_n - O_mO_{n+1} = \begin{cases} -\frac{1}{p\sqrt{p^2-4}}\left(\alpha^m\beta^n - \alpha^n\beta^m\right) & , \text{ if } p^2 \neq 4 \\ -2^{-m-n+1}\left(m-n\right)\frac{1}{p^2} & , \text{ if } p^2 = 4 \end{cases}.$$

(b) *(Gelin-Cesàro's identity)*

$$O_{n+2}O_{n+1}O_{n-1}O_{n-2} - O_n^4 = \begin{cases} \frac{1}{p^{2n}\left(p^2-4\right)}((p^4-2p^2+2)\alpha^n\beta^n - (p^2+1)(\alpha^{2n}+\beta^{2n})) & , \text{ if } p^2 \neq 4 \\ -2^{-4n+4}\left(5n^2-4\right)\frac{1}{p^4} & , \text{ if } p^2 = 4 \end{cases}$$

5 ON THE RECURRENCE PROPERTIES OF GENERALIZED P-ORESME SEQUENCE

Taking $r=1, s=-\frac{1}{p^2}$ in Theorem 1.1 (a) and (b), we obtain the following Proposition.

Proposition 5.1. *For $n\in\mathbb{Z}$, generalized Oresme numbers (the case $r=1, s=-\frac{1}{p^2}$) have the following identity:*

$$\begin{aligned} W_{-n} &= -p^{2n}(W_n - H_nW_0) \\ &= \frac{p^{2n}}{p^2W_1^2+W_0^2-p^2W_0W_1}(p^2(2W_1-W_0)W_0W_{n+1}-(p^2W_1^2-W_0^2)W_n). \end{aligned}$$

From the above Proposition, we have the following corollary which gives the connection between the special cases of generalized p-Oresme sequence at the positive index and the negative index: for modified p-Oresme, p-Oresme-Lucas and p-Oresme numbers: take $W_n=G_n$ with $G_0=0, G_1=1,$ take $W_n=H_n$ with $H_0=2, H_1=1$ and $W_n=O_n$ with $O_0=0, O_1=\frac{1}{p}$, respectively. Note that in this case $H_n=H_n$.

Corollary 5.2. *For $n\in\mathbb{Z}$, we have the following recurrence relations:*

(a) *modified p-Oresme sequence:*

$$\begin{aligned} G_{-n} &= -p^{2n}G_n \\ &= \begin{cases} \frac{-p^{n+1}}{2^n\sqrt{p^2-4}}\left(\left(p+\sqrt{p^2-4}\right)^n-\left(p-\sqrt{p^2-4}\right)^n\right) & , \text{ if } p^2\neq 4 \\ -\frac{n}{2^{n-1}}p^{2n} & , \text{ if } p^2=4 \end{cases}. \end{aligned}$$

(b) *p-Oresme-Lucas sequence:*

$$\begin{aligned} H_{-n} &= p^{2n}H_n \\ &= \frac{p^n}{2^n}\left(\left(p+\sqrt{p^2-4}\right)^n+\left(p-\sqrt{p^2-4}\right)^n\right). \end{aligned}$$

(c) *p-Oresme sequence:*

$$\begin{aligned} O_{-n} &= -p^{2n}O_n \\ &= \begin{cases} \frac{-p^{n}}{2^n\sqrt{p^2-4}}\left(\left(p+\sqrt{p^2-4}\right)^n-\left(p-\sqrt{p^2-4}\right)^n\right) & , \text{ if } p^2\neq 4 \\ -\frac{n}{2^{n-1}}p^{2n-1} & , \text{ if } p^2=4 \end{cases}. \end{aligned}$$

6 THE SUM FORMULA $\sum_{k=0}^{n} x^kW_{mk+j}$

In this section, we present sum formulas of generalized p-Oresme numbers. As special cases of m and j in Theorem 1.2, we obtain the following proposition.

Proposition 6.1. *For generalized p-Oresme numbers, we have the following sum formulas:*

(a) $(m=1, j=0)$

If $-p^2x+p^2+x^2\neq 0$, i.e., $x\neq\frac{1}{2}p\left(p+\sqrt{p^2-4}\right)$, $x\neq\frac{1}{2}p\left(p-\sqrt{p^2-4}\right)$, then

$$\sum_{k=0}^{n} x^kW_k = \frac{(x-p^2)x^{n+1}W_n+x^{n+1}W_{n-1}+p^2W_0+p^2(W_1-W_0)x}{(-p^2x+p^2+x^2)},$$

and
if $-p^2x+p^2+x^2=0$, *i.e.,* $x=\frac{1}{2}p\left(p+\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2}p\left(p-\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kW_k=\frac{\left(\left(x-p^2\right)n+2x-p^2\right)x^nW_n+(n+1)x^nW_{n-1}+p^2(W_1-W_0)}{(2x-p^2)}.$$

(b) $(m=2, j=0)$
If $2p^2x-p^4x+p^4+x^2\neq 0$, *i.e.,* $x\neq\frac{1}{2}p^2\left(p^2-2+p\sqrt{p^2-4}\right)$, $x\neq\frac{1}{2}p^2\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kW_{2k}=\frac{(x+2p^2-p^4)x^{n+1}W_{2n}+x^{n+1}W_{2n-2}+p^2(p^2xW_1+(x-p^2x+p^2)W_0)}{(x^2+2p^2x-p^4x+p^4)},$$

and
if $2p^2x-p^4x+p^4+x^2=0$, *i.e.,* $x=\frac{1}{2}p^2\left(p^2-2+p\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2}p^2\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kW_{2k}=\frac{((x+2p^2-p^4)n+2x+2p^2-p^4)x^nW_{2n}+(n+1)x^nW_{2n-2}+p^2\left(p^2W_1-W_0(p^2-1)\right)}{(2x+2p^2-p^4)}.$$

(c) $(m=2, j=1)$
If $2p^2x-p^4x+p^4+x^2\neq 0$, *i.e.,* $x\neq\frac{1}{2}p^2\left(p^2-2+p\sqrt{p^2-4}\right)$, $x\neq\frac{1}{2}p^2\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kW_{2k+1}=\frac{(x+2p^2-p^4)x^{n+1}W_{2n+1}+x^{n+1}W_{2n-1}+p^2((x+p^2)W_1-xW_0)}{(x^2+2p^2x-p^4x+p^4)},$$

and
if $2p^2x-p^4x+p^4+x^2=0$, *i.e.,* $x=\frac{1}{2}p^2\left(p^2-2+p\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2}p^2\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kW_{2k+1}=\frac{((x+2p^2-p^4)n+2x+2p^2-p^4)x^nW_{2n+1}+(n+1)x^nW_{2n-1}+p^2(W_1-W_0)}{(2x+2p^2-p^4)}.$$

(d) $(m=-1, j=0)$
If $-p^2x+p^2x^2+1\neq 0$, *i.e.,* $x\neq\frac{1}{2p}\left(p+\sqrt{p^2-4}\right)$, $x\neq\frac{1}{2p}\left(p-\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kW_{-k}=\frac{p^2x^{n+1}W_{-n+1}+p^2(x-1)x^{n+1}W_{-n}-p^2xW_1+W_0}{-p^2x+p^2x^2+1},$$

and
if $-p^2x+p^2x^2+1=0$, *i.e.,* $x=\frac{1}{2p}\left(p+\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2p}\left(p-\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kW_{-k}=\frac{p^2(n+1)x^nW_{-n+1}+p^2\left((x-1)n+2x-1\right)x^nW_{-n}-p^2W_1}{2p^2x-b_1}.$$

(e) $(m=-2, j=0)$
If $2p^2x-p^4x+p^4x^2+1\neq 0$, *i.e.,* $x\neq\frac{1}{2p^2}\left(p^2-2+p\sqrt{p^2-4}\right)$, $x\neq\frac{1}{2p^2}\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kW_{-2k}=\frac{p^4x^{n+1}W_{-2n+2}+p^2(p^2x-p^2+2)x^{n+1}W_{-2n}-p^4xW_1+(p^2x+1)W_0}{p^4x^2-p^4x+2p^2x+1},$$

and
if $2p^2x-p^4x+p^4x^2+1=0$, *i.e.,* $x=\frac{1}{2p^2}\left(p^2-2+p\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2p^2}\left(p^2-2-p\sqrt{p^2-4}\right)$,
then

$$\sum_{k=0}^{n}x^kW_{-2k}=\frac{p^2(n+1)x^nW_{-2n+2}+((p^2x-p^2+2)n+2p^2x-p^2+2)x^nW_{-2n}-p^2W_1+W_0}{(2p^2x-p^2+2)}.$$

(f) $(m=-2,j=1)$
If $2p^2x-p^4x+p^4x^2+1\neq 0$, *i.e.,* $x\neq\frac{1}{2p^2}\left(p^2-2+p\sqrt{p^2-4}\right)$, $x\neq\frac{1}{2p^2}\left(p^2-2-p\sqrt{p^2-4}\right)$,
then

$$\sum_{k=0}^{n}x^kW_{-2k+1}=\frac{p^4x^{n+1}W_{-2n+3}+p^2(p^2x-p^2+2)x^{n+1}W_{-2n+1}+(p^2x-p^4x+1)W_1+p^2xW_0}{p^4x^2-p^4x+2p^2x+1},$$

and
if $2p^2x-p^4x+p^4x^2+1=0$, *i.e.,* $x=\frac{1}{2p^2}\left(p^2-2+p\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2p^2}\left(p^2-2-p\sqrt{p^2-4}\right)$,
then

$$\sum_{k=0}^{n}x^kW_{-2k+1}=\frac{p^2(n+1)x^nW_{-2n+3}+((p^2x-p^2+2)n+2p^2x-p^2+2)x^nW_{-2n+1}-W_1(p^2-1)+W_0}{(2p^2x-p^2+2)}.$$

Proof. Take $r=1,s=-\dfrac{1}{p^2}$ and $H_n=H_n$ in Theorem 1.2. Note that the sum formulas for the case $p=2$ is given in Soykan [6]. □

From the above proposition, we have the following corollary which gives sum formulas of modified p-Oresme numbers (take $W_n=G_n$ with $G_0=0,G_1=1$).

Corollary 6.2. *For* $n\geq 0$, *modified p-Oresme numbers have the following properties:*

(a) $(m=1,j=0)$
If $-p^2x+p^2+x^2\neq 0$, *i.e.,* $x\neq\frac{1}{2}p\left(p+\sqrt{p^2-4}\right)$, $x\neq\frac{1}{2}p\left(p-\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kG_k=\frac{(x-p^2)x^{n+1}G_n+x^{n+1}G_{n-1}+p^2x}{(-p^2x+p^2+x^2)},$$

and
if $-p^2x+p^2+x^2=0$, *i.e.,* $x=\frac{1}{2}p\left(p+\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2}p\left(p-\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kG_k=\frac{((x-p^2)\,n+2x-p^2)x^nG_n+(n+1)x^nG_{n-1}+p^2}{(2x-p^2)}.$$

(b) $(m=2,j=0)$
If $2p^2x-p^4x+p^4+x^2\neq 0$, *i.e.,* $x\neq\frac{1}{2}p^2\left(p^2-2+p\sqrt{p^2-4}\right)$, $x\neq\frac{1}{2}p^2\left(p^2-2-p\sqrt{p^2-4}\right)$,
then

$$\sum_{k=0}^{n}x^kG_{2k}=\frac{(x+2p^2-p^4)x^{n+1}G_{2n}+x^{n+1}G_{2n-2}+p^4x}{(x^2+2p^2x-p^4x+p^4)},$$

and
if $2p^2x-p^4x+p^4+x^2=0$, *i.e.,* $x=\frac{1}{2}p^2\left(p^2-2+p\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2}p^2\left(p^2-2-p\sqrt{p^2-4}\right)$,
then

$$\sum_{k=0}^{n}x^kG_{2k}=\frac{((x+2p^2-p^4)n+2x+2p^2-p^4)x^nG_{2n}+(n+1)x^nG_{2n-2}+p^4}{(2x+2p^2-p^4)}.$$

(c) $(m=2, j=1)$

If $2p^2x-p^4x+p^4+x^2\neq 0$, *i.e.,* $x\neq \frac{1}{2}p^2\left(p^2-2+p\sqrt{p^2-4}\right), x\neq \frac{1}{2}p^2\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kG_{2k+1}=\frac{(x+2p^2-p^4)x^{n+1}G_{2n+1}+x^{n+1}G_{2n-1}+p^2(p^2+x)}{(x^2+2p^2x-p^4x+p^4)},$$

and

if $2p^2x-p^4x+p^4+x^2=0$, *i.e.,* $x=\frac{1}{2}p^2\left(p^2-2+p\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2}p^2\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kG_{2k+1}=\frac{((x+2p^2-p^4)n+2x+2p^2-p^4)x^nG_{2n+1}+(n+1)x^nG_{2n-1}+p^2}{(2x+2p^2-p^4)}.$$

(d) $(m=-1, j=0)$

If $-p^2x+p^2x^2+1\neq 0$, *i.e.,* $x\neq \frac{1}{2p}\left(p+\sqrt{p^2-4}\right), x\neq \frac{1}{2p}\left(p-\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kG_{-k}=\frac{p^2x^{n+1}G_{-n+1}+p^2(x-1)x^{n+1}G_{-n}-p^2x}{-p^2x+p^2x^2+1},$$

and

if $-p^2x+p^2x^2+1=0$, *i.e.,* $x=\frac{1}{2p}\left(p+\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2p}\left(p-\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kG_{-k}=\frac{p^2(n+1)x^nG_{-n+1}+p^2\left((x-1)n+2x-1\right)x^nG_{-n}-p^2}{2p^2x-b_1}.$$

(e) $(m=-2, j=0)$

If $2p^2x-p^4x+p^4x^2+1\neq 0$, *i.e.,* $x\neq \frac{1}{2p^2}\left(p^2-2+p\sqrt{p^2-4}\right), x\neq \frac{1}{2p^2}\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kG_{-2k}=\frac{p^4x^{n+1}G_{-2n+2}+p^2(p^2x-p^2+2)x^{n+1}G_{-2n}-p^4x}{p^4x^2-p^4x+2p^2x+1},$$

and

if $2p^2x-p^4x+p^4x^2+1=0$, *i.e.,* $x=\frac{1}{2p^2}\left(p^2-2+p\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2p^2}\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kG_{-2k}=\frac{p^2(n+1)x^nG_{-2n+2}+((p^2x-p^2+2)n+2p^2x-p^2+2)x^nG_{-2n}-p^2}{(2p^2x-p^2+2)}.$$

(f) $(m=-2, j=1)$

If $2p^2x-p^4x+p^4x^2+1\neq 0$, *i.e.,* $x\neq \frac{1}{2p^2}\left(p^2-2+p\sqrt{p^2-4}\right), x\neq \frac{1}{2p^2}\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kG_{-2k+1}=\frac{p^4x^{n+1}G_{-2n+3}+p^2(p^2x-p^2+2)x^{n+1}G_{-2n+1}+(p^2x-p^4x+1)}{p^4x^2-p^4x+2p^2x+1},$$

and

if $2p^2x-p^4x+p^4x^2+1=0$, *i.e.,* $x=\frac{1}{2p^2}\left(p^2-2+p\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2p^2}\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kG_{-2k+1}=\frac{p^2(n+1)x^nG_{-2n+3}+((p^2x-p^2+2)n+2p^2x-p^2+2)x^nG_{-2n+1}+1-p^2}{(2p^2x-p^2+2)}.$$

Taking $W_n = H_n$ with $H_0 = 2, H_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of p-Oresme-Lucas numbers.

Corollary 6.3. *For $n \geq 0$, p-Oresme-Lucas numbers have the following properties:*

(a) $(m = 1, j = 0)$

If $-p^2x + p^2 + x^2 \neq 0$, *i.e.,* $x \neq \frac{1}{2}p\left(p + \sqrt{p^2 - 4}\right)$, $x \neq \frac{1}{2}p\left(p - \sqrt{p^2 - 4}\right)$, *then*

$$\sum_{k=0}^{n} x^k H_k = \frac{(x - p^2)x^{n+1}H_n + x^{n+1}H_{n-1} + 2p^2 - p^2x}{(-p^2x + p^2 + x^2)},$$

and

if $-p^2x + p^2 + x^2 = 0$, *i.e.,* $x = \frac{1}{2}p\left(p + \sqrt{p^2 - 4}\right)$ *or* $x = \frac{1}{2}p\left(p - \sqrt{p^2 - 4}\right)$, *then*

$$\sum_{k=0}^{n} x^k H_k = \frac{\left(\left(x - p^2\right)n + 2x - p^2\right)x^nH_n + (n+1)x^nH_{n-1} - p^2}{(2x - p^2)}.$$

(b) $(m = 2, j = 0)$

If $2p^2x - p^4x + p^4 + x^2 \neq 0$, *i.e.,* $x \neq \frac{1}{2}p^2\left(p^2 - 2 + p\sqrt{p^2 - 4}\right)$, $x \neq \frac{1}{2}p^2\left(p^2 - 2 - p\sqrt{p^2 - 4}\right)$, *then*

$$\sum_{k=0}^{n} x^k H_{2k} = \frac{(x + 2p^2 - p^4)x^{n+1}H_{2n} + x^{n+1}H_{2n-2} + p^2(2x - p^2x + 2p^2)}{(x^2 + 2p^2x - p^4x + p^4)},$$

and

if $2p^2x - p^4x + p^4 + x^2 = 0$, *i.e.,* $x = \frac{1}{2}p^2\left(p^2 - 2 + p\sqrt{p^2 - 4}\right)$ *or* $x = \frac{1}{2}p^2\left(p^2 - 2 - p\sqrt{p^2 - 4}\right)$, *then*

$$\sum_{k=0}^{n} x^k H_{2k} = \frac{((x + 2p^2 - p^4)n + 2x + 2p^2 - p^4)x^nH_{2n} + (n+1)x^nH_{2n-2} - p^2(p^2 - 2)}{(2x + 2p^2 - p^4)}.$$

(c) $(m = 2, j = 1)$

If $2p^2x - p^4x + p^4 + x^2 \neq 0$, *i.e.,* $x \neq \frac{1}{2}p^2\left(p^2 - 2 + p\sqrt{p^2 - 4}\right)$, $x \neq \frac{1}{2}p^2\left(p^2 - 2 - p\sqrt{p^2 - 4}\right)$, *then*

$$\sum_{k=0}^{n} x^k H_{2k+1} = \frac{(x + 2p^2 - p^4)x^{n+1}H_{2n+1} + x^{n+1}H_{2n-1} - p^2(x - p^2)}{(x^2 + 2p^2x - p^4x + p^4)},$$

and

if $2p^2x - p^4x + p^4 + x^2 = 0$, *i.e.,* $x = \frac{1}{2}p^2\left(p^2 - 2 + p\sqrt{p^2 - 4}\right)$ *or* $x = \frac{1}{2}p^2\left(p^2 - 2 - p\sqrt{p^2 - 4}\right)$, *then*

$$\sum_{k=0}^{n} x^k H_{2k+1} = \frac{((x + 2p^2 - p^4)n + 2x + 2p^2 - p^4)x^nH_{2n+1} + (n+1)x^nH_{2n-1} - p^2}{(2x + 2p^2 - p^4)}.$$

(d) $(m = -1, j = 0)$

If $-p^2x + p^2x^2 + 1 \neq 0$, *i.e.,* $x \neq \frac{1}{2p}\left(p + \sqrt{p^2 - 4}\right)$, $x \neq \frac{1}{2p}\left(p - \sqrt{p^2 - 4}\right)$, *then*

$$\sum_{k=0}^{n} x^k H_{-k} = \frac{p^2x^{n+1}H_{-n+1} + p^2(x-1)x^{n+1}H_{-n} + 2 - p^2x}{-p^2x + p^2x^2 + 1},$$

and

if $-p^2x+p^2x^2+1=0$, *i.e.,* $x=\frac{1}{2p}\left(p+\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2p}\left(p-\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kH_{-k}=\frac{p^2(n+1)x^nH_{-n+1}+p^2\left((x-1)n+2x-1\right)x^nH_{-n}-p^2}{2p^2x-b_1}.$$

(e) $(m=-2,\, j=0)$

If $2p^2x-p^4x+p^4x^2+1\neq 0$, *i.e.,* $x\neq\frac{1}{2p^2}\left(p^2-2+p\sqrt{p^2-4}\right)$, $x\neq\frac{1}{2p^2}\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kH_{-2k}=\frac{p^4x^{n+1}H_{-2n+2}+p^2(p^2x-p^2+2)x^{n+1}H_{-2n}-xp^4+2xp^2+2}{p^4x^2-p^4x+2p^2x+1},$$

and
if $2p^2x-p^4x+p^4x^2+1=0$, *i.e.,* $x=\frac{1}{2p^2}\left(p^2-2+p\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2p^2}\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kH_{-2k}=\frac{p^2(n+1)x^nH_{-2n+2}+((p^2x-p^2+2)n+2p^2x-p^2+2)x^nH_{-2n}+2-p^2}{(2p^2x-p^2+2)}.$$

(f) $(m=-2,\, j=1)$

If $2p^2x-p^4x+p^4x^2+1\neq 0$, *i.e.,* $x\neq\frac{1}{2p^2}\left(p^2-2+p\sqrt{p^2-4}\right)$, $x\neq\frac{1}{2p^2}\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kH_{-2k+1}=\frac{p^4x^{n+1}H_{-2n+3}+p^2(p^2x-p^2+2)x^{n+1}H_{-2n+1}-xp^4+3xp^2+1}{p^4x^2-p^4x+2p^2x+1},$$

and
if $2p^2x-p^4x+p^4x^2+1=0$, *i.e.,* $x=\frac{1}{2p^2}\left(p^2-2+p\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2p^2}\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kH_{-2k+1}=\frac{p^2(n+1)x^nH_{-2n+3}+((p^2x-p^2+2)n+2p^2x-p^2+2)x^nH_{-2n+1}+3-p^2}{(2p^2x-p^2+2)}.$$

From the above proposition, we have the following corollary which gives sum formulas of p-Oresme numbers (take $W_n=O_n$ with $O_0=0, O_1=\frac{1}{p}$).

Corollary 6.4. *For* $n\geq 0$, *p-Oresme numbers have the following properties:*

(a) $(m=1,\, j=0)$

If $-p^2x+p^2+x^2\neq 0$, *i.e.,* $x\neq\frac{1}{2}p\left(p+\sqrt{p^2-4}\right)$, $x\neq\frac{1}{2}p\left(p-\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kO_k=\frac{(x-p^2)x^{n+1}O_n+x^{n+1}O_{n-1}+px}{(-p^2x+p^2+x^2)},$$

and
if $-p^2x+p^2+x^2=0$, *i.e.,* $x=\frac{1}{2}p\left(p+\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2}p\left(p-\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n}x^kO_k=\frac{(\left(x-p^2\right)n+2x-p^2)x^nO_n+(n+1)x^nO_{n-1}+p}{(2x-p^2)}.$$

(b) $(m=2, j=0)$

If $2p^2x-p^4x+p^4+x^2 \neq 0$, *i.e.,* $x \neq \frac{1}{2}p^2\left(p^2-2+p\sqrt{p^2-4}\right), x \neq \frac{1}{2}p^2\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n} x^k O_{2k} = \frac{(x+2p^2-p^4)x^{n+1}O_{2n}+x^{n+1}O_{2n-2}+p^3x}{(x^2+2p^2x-p^4x+p^4)},$$

and

if $2p^2x-p^4x+p^4+x^2=0$, *i.e.,* $x=\frac{1}{2}p^2\left(p^2-2+p\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2}p^2\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n} x^k O_{2k} = \frac{((x+2p^2-p^4)n+2x+2p^2-p^4)x^nO_{2n}+(n+1)x^nO_{2n-2}+p^3}{(2x+2p^2-p^4)}.$$

(c) $(m=2, j=1)$

If $2p^2x-p^4x+p^4+x^2 \neq 0$, *i.e.,* $x \neq \frac{1}{2}p^2\left(p^2-2+p\sqrt{p^2-4}\right), x \neq \frac{1}{2}p^2\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n} x^k O_{2k+1} = \frac{(x+2p^2-p^4)x^{n+1}O_{2n+1}+x^{n+1}O_{2n-1}+p(p^2+x)}{(x^2+2p^2x-p^4x+p^4)},$$

and

if $2p^2x-p^4x+p^4+x^2=0$, *i.e.,* $x=\frac{1}{2}p^2\left(p^2-2+p\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2}p^2\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n} x^k O_{2k+1} = \frac{((x+2p^2-p^4)n+2x+2p^2-p^4)x^nO_{2n+1}+(n+1)x^nO_{2n-1}+p}{(2x+2p^2-p^4)}.$$

(d) $(m=-1, j=0)$

If $-p^2x+p^2x^2+1 \neq 0$, *i.e.,* $x \neq \frac{1}{2p}\left(p+\sqrt{p^2-4}\right), x \neq \frac{1}{2p}\left(p-\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n} x^k O_{-k} = \frac{p^2x^{n+1}O_{-n+1}+p^2(x-1)x^{n+1}O_{-n}-px}{-p^2x+p^2x^2+1},$$

and

if $-p^2x+p^2x^2+1=0$, *i.e.,* $x=\frac{1}{2p}\left(p+\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2p}\left(p-\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n} x^k O_{-k} = \frac{p^2(n+1)x^nO_{-n+1}+p^2\left((x-1)n+2x-1\right)x^nO_{-n}-p}{2p^2x-b_1}.$$

(e) $(m=-2, j=0)$

If $2p^2x-p^4x+p^4x^2+1 \neq 0$, *i.e.,* $x \neq \frac{1}{2p^2}\left(p^2-2+p\sqrt{p^2-4}\right), x \neq \frac{1}{2p^2}\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n} x^k O_{-2k} = \frac{p^4x^{n+1}O_{-2n+2}+p^2(p^2x-p^2+2)x^{n+1}O_{-2n}-p^3x}{p^4x^2-p^4x+2p^2x+1},$$

and

if $2p^2x-p^4x+p^4x^2+1=0$, *i.e.,* $x=\frac{1}{2p^2}\left(p^2-2+p\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2p^2}\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n} x^k O_{-2k} = \frac{p^2(n+1)x^nO_{-2n+2}+((p^2x-p^2+2)n+2p^2x-p^2+2)x^nO_{-2n}-p}{(2p^2x-p^2+2)}.$$

(f) $(m=-2, j=1)$

If $2p^2x-p^4x+p^4x^2+1\neq 0$, *i.e.,* $x\neq \frac{1}{2p^2}\left(p^2-2+p\sqrt{p^2-4}\right), x\neq \frac{1}{2p^2}\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n} x^k O_{-2k+1} = \frac{p^4x^{n+1}O_{-2n+3}+p^2(p^2x-p^2+2)x^{n+1}O_{-2n+1}+\frac{1}{p}(-xp^4+xp^2+1)}{p^4x^2-p^4x+2p^2x+1},$$

and

if $2p^2x-p^4x+p^4x^2+1=0$, *i.e.,* $x=\frac{1}{2p^2}\left(p^2-2+p\sqrt{p^2-4}\right)$ *or* $x=\frac{1}{2p^2}\left(p^2-2-p\sqrt{p^2-4}\right)$, *then*

$$\sum_{k=0}^{n} x^k O_{-2k+1} = \frac{p^2(n+1)x^nO_{-2n+3}+((p^2x-p^2+2)n+2p^2x-p^2+2)x^nO_{-2n+1}-\frac{1}{p}(p^2-1)}{(2p^2x-p^2+2)}.$$

Taking $x=1$ in the last three corollaries we get the following corollary.

Corollary 6.5. *For* $n\geq 0$, *modified p-Oresme numbers, p-Oresme-Lucas numbers and p-Oresme numbers have the following properties:*

1. *modified p-Oresme numbers:*
   - **(a)** $\sum_{k=0}^{n} G_k = -\left(p^2-1\right)G_n+G_{n-1}+p^2.$
   - **(b)** $\sum_{k=0}^{n} G_{2k} = \dfrac{((-p^4+2p^2+1)G_{2n}+G_{2n-2}+p^4)}{2p^2+1}.$
   - **(c)** $\sum_{k=0}^{n} G_{2k+1} = \dfrac{((-p^4+2p^2+1)G_{2n+1}+G_{2n-1}+p^2\left(p^2+1\right))}{2p^2+1}.$
   - **(d)** $\sum_{k=0}^{n} G_{-k} = p^2G_{-n+1}-p^2.$
   - **(e)** $\sum_{k=0}^{n} G_{-2k} = \dfrac{(p^4G_{-2n+2}+2p^2G_{-2n}-p^4)}{2p^2+1}.$
   - **(f)** $\sum_{k=0}^{n} G_{-2k+1} = \dfrac{(p^4G_{-2n+3}+2p^2G_{-2n+1}+p^2-p^4+1)}{2p^2+1}.$
2. *p-Oresme-Lucas numbers:*
   - **(a)** $\sum_{k=0}^{n} H_k = -(p^2-1)H_n+H_{n-1}+p^2.$
   - **(b)** $\sum_{k=0}^{n} H_{2k} = \dfrac{((-p^4+2p^2+1)H_{2n}+H_{2n-2}+p^2(p^2+2))}{2p^2+1}.$
   - **(c)** $\sum_{k=0}^{n} H_{2k+1} = \dfrac{((-p^4+2p^2+1)H_{2n+1}+H_{2n-1}-p^2(1-p^2))}{2p^2+1}.$
   - **(d)** $\sum_{k=0}^{n} H_{-k} = p^2H_{-n+1}-p^2+2.$
   - **(e)** $\sum_{k=0}^{n} H_{-2k} = \dfrac{(p^4H_{-2n+2}+2p^2H_{-2n}-p^4+2p^2+2)}{2p^2+1}.$
   - **(f)** $\sum_{k=0}^{n} H_{-2k+1} = \dfrac{(p^4H_{-2n+3}+2p^2H_{-2n+1}-p^4+3p^2+1)}{2p^2+1}.$
3. *p-Oresme numbers:*
   - **(a)** $\sum_{k=0}^{n} O_k = -(p^2-1)O_n+O_{n-1}+p.$

(b) $\sum_{k=0}^{n} O_{2k} = \dfrac{((-p^4+2p^2+1)O_{2n}+O_{2n-2}+p^3)}{2p^2+1}.$

(c) $\sum_{k=0}^{n} O_{2k+1} = \dfrac{((-p^4+2p^2+1)O_{2n+1}+O_{2n-1}+p(p^2+1))}{2p^2+1}$

(d) $\sum_{k=0}^{n} O_{-k} = p^2 O_{-n+1} - p.$

(e) $\sum_{k=0}^{n} O_{-2k} = \dfrac{(p^4 O_{-2n+2}+2p^2 O_{-2n}-p^3)}{2p^2+1}.$

(f) $\sum_{k=0}^{n} O_{-2k+1} = \dfrac{(p^4 O_{-2n+3}+2p^2 O_{-2n+1}+\frac{1}{p}(-p^4+p^2+1))}{2p^2+1}.$

7 MATRICES RELATED WITH GENERALIZED P-ORESME NUMBERS

We define the square matrix A of order 2 as:

$$A = \begin{pmatrix} 1 & -\frac{1}{p^2} \\ 1 & 0 \end{pmatrix}$$

such that $\det A = \dfrac{1}{p^2}$. Then, we have

$$\begin{pmatrix} W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{p^2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W_n \\ W_{n-1} \end{pmatrix} \tag{7.1}$$

and

$$\begin{pmatrix} W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{p^2} \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_1 \\ W_0 \end{pmatrix}.$$

If we take $W_n = G_n$ in (7.1) we have

$$\begin{pmatrix} G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{p^2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} G_n \\ G_{n-1} \end{pmatrix}. \tag{7.2}$$

We also define

$$B_n = \begin{pmatrix} G_{n+1} & -\frac{1}{p^2}G_n \\ G_n & -\frac{1}{p^2}G_{n-1} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} W_{n+1} & -\frac{1}{p^2}W_n \\ W_n & -\frac{1}{p^2}W_{n-1} \end{pmatrix}.$$

Theorem 7.1. *For all integers $m, n,$ we have*

(a) $B_n = A^n$

(b) $C_1 A^n = A^n C_1$

(c) $C_{n+m} = C_n B_m = B_m C_n.$

Proof. Take $r=1, s=-\dfrac{1}{p^2}$ in Soykan [[11], Theorem 5.1.]. □

Corollary 7.2. *For all integers n and $0 \neq p \in \mathbb{R}$, we have the following formulas for the modified p-Oresme, p-Oresme-Lucas and p-Oresme numbers.*

(a) *Modified p-Oresme Numbers.*

$$A^n = \begin{pmatrix} 1 & -\dfrac{1}{p^2} \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & -\dfrac{1}{p^2}G_n \\ G_n & -\dfrac{1}{p^2}G_{n-1} \end{pmatrix}.$$

(b) *p-Oresme-Lucas Numbers. If $p^2 \neq 4$ then*

$$A^n = \begin{pmatrix} 1 & -\dfrac{1}{p^2} \\ 1 & 0 \end{pmatrix}^n = \tfrac{1}{p^2-4}\begin{pmatrix} p^2(2H_{n+2}-H_{n+1}) & -(2H_{n+1}-H_n) \\ p^2(2H_{n+1}-H_n) & -(2H_n-H_{n-1}) \end{pmatrix},$$

and if $p^2 = 4$ then

$$A^n = \begin{pmatrix} 1 & -\dfrac{1}{4} \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} (n+1)H_{n+1} & -\frac{1}{4}nH_n \\ nH_n & -\frac{1}{4}(n-1)H_{n-1} \end{pmatrix}.$$

(c) *p-Oresme Numbers.*

$$A^n = \begin{pmatrix} 1 & -\dfrac{1}{p^2} \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} pO_{n+1} & -\dfrac{1}{p}O_n \\ pO_n & -\dfrac{1}{p}O_{n-1} \end{pmatrix}.$$

Proof.

(a) It is given in Theorem 7.1 (a).

(b) Note that if $p^2 \neq 4$ then, from Lemma 4.4, we have

$$(p^2-4)G_n = 2p^2H_{n+1} - p^2H_n$$

and if $p^2 = 4$ then, from (2.8),we have

$$G_n = nH_n.$$

Using the last two equations and (a), we get required result.

(c) Note that, from (2.7) or Lemma 4.5, we have

$$G_n = pO_n.$$

Using the last equation and (a), we get required result. □

Theorem 7.3. *For all integers m, n, we have*

$$W_{n+m} = W_nG_{m+1} - \frac{1}{p^2}W_{n-1}G_m. \tag{7.3}$$

Proof. Take $r=1, s=-\frac{1}{p^2}$ in Soykan [[11], Theorem 5.2.]. □
By Lemma 4.1, we know that

$$(-p^2W_1^2 - W_0^2 + p^2W_0W_1)G_m = p^2W_0W_{m+1} - p^2W_1W_m,$$

so (7.3) can be written in the following form

$$(-p^2W_1^2 - W_0^2 + p^2W_0W_1)W_{n+m} = W_n(p^2(W_0-W_1)W_{m+1} - W_0W_m) + W_{n-1}(-W_0W_{m+1} + W_1W_m).$$

Corollary 7.4. *For all integers* $m, n,$ *we have*

$$\begin{aligned} G_{n+m} &= G_n G_{m+1} - \frac{1}{p^2} G_{n-1} G_m, \\ H_{n+m} &= H_n G_{m+1} - \frac{1}{p^2} H_{n-1} G_m, \\ O_{n+m} &= O_n G_{m+1} - \frac{1}{p^2} O_{n-1} G_m, \end{aligned}$$

and

$$O_{n+m} = \frac{1}{p}(p^2 O_n O_{m+1} - O_m O_{n-1}).$$

8 CONCLUSION

Sequences have been fascinating topic for mathematicians for centuries. There are so many studies in the literature that concern about special second order recurrence sequences such as Fibonacci and Lucas sequences, see for example [15],[16],[17],[18],[19]. In this paper, we obtain some fundamental properties of generalized p-Oresme numbers. We can summarize the sections as follows:

- In section 1, we give some background about generalized Fibonacci numbers and present a short history of Oresme numbers.
- In section 2, we define generalized p-Oresme sequence and then the generating functions and the Binet's formulas have been given.
- In section 3, Simson formula of generalized p-Oresme numbers are presented.
- In section 4, we obtain some identities of generalized p-Oresme, modified p-Oresme, p-Oresme-Lucas and p-Oresme numbers.
- In section 5, we consider generalized p-Oresme sequence at negative indices and construct the relationship between the sequence and itself at positive indices. This illustrates the recurrence property of the sequence at the negative index. Meanwhile, this connection holds for all integers.
- In section 6, we have written sum identities in terms of the generalized p-Oresme sequence, and then we have presented the formulas as special cases the corresponding identity for the modified p-Oresme, p-Oresme-Lucas and p-Oresme sequences. All the listed identities in the proposition and corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.
- In section 7, we give matrices related with these sequences (generalized p-Oresme, modified p-Oresme, p-Oresme-Lucas and p-Oresme sequences).

COMPETING INTERESTS

Author has declared that no competing interests exist.

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