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On Orthogonal Double Covers of Circulant Graphs

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Abstract

Let X be a graph on n vertices and let $B = \{P(x) : x \in V(X)\}$ be a collection of n subgraphs of X, one for each vertex, B is an orthogonal double cover (ODC) of X if every edge of X occurs in exactly two members of B and any two members share an edge whenever the corresponding vertices are adjacent in X and share no edges whenever the corresponding vertices are nonadjacent in X. The main question is: given the pair (X, G), is there an ODC of X by G? An obvious necessary condition is that X is a regular. In this paper, we are almost exclusively concerned with the starter maps of the orthogonal double covers of cayley graphs and using this method to construct ODCs by a complete bipartite graph, a complete tripartite graph, caterpillar, and a connected union of a cycle and a star whose center vertex belongs to that cycle.

Keywords: Cayley graph; Graph decomposition; Orthogonal double cover; Symmetric starter

1 Introduction

Let X and G be graphs, such that G is isomorphic to a subgraph of X. An ODC of X by G is a collection $B = \{P(x) : x \in V(X)\}$ of subgraphs of X, all isomorphic to G, such that

(i) every edge of X occurs in exactly two members of B and (ii) $P(x)$ and $P(y)$ share an edge if and only if x and y are adjacent in X. The elements of B will be called pages.

This concept is a natural generalization of earlier definitions of an ODC for complete and complete bipartite graphs, that have been studied extensively (see

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the survey [1]). The main question is: given the pair (X, G), is there an ODC of X by G? An obvious necessary condition is that X is a regular. In this case, the answer is surely affirmative when G is a star: such ODCs will be called trivial.

An effective technique to construct ODCs in the above cases was based on the idea of translate a given subgraph of G by a group acting on $V(X)$. Thus, we want to tackle this way this more general definition of ODC in the case of Cayley graphs. In [2], Scapellato et al deals with Cayley graphs of degree 2 and 3. and offers some insights on the case on ODC of Cayley graphs on cyclic groups. Multiplicative notation for groups is handy and will be used as default. Here, in this paper we will switch to additive notation when groups of residue classes are involved.

Note that an edge of a graph will often be identified with one of its arcs, so we will write $(x,\ y)$ instead of $\{x, y\}$.

An automorphism φ of B is a map from $V(X)$ to itself such that $\varphi(P(x)) = P(\varphi(x))$ for all x. If a coloring is assigned to edges of X, the automorphism φ will be called colour-preserving if whenever $(x,\ y) \in E(X)$ the edges $(x,\ y)$ and $(\varphi(x),\ \varphi(y))$ have the same colour. An ODC B of X is cyclic (CODC) if the cyclic group of order $|V(X)|$ is a subgroup of the automorphism group of B, the set of all automorphisms of B. Note that in this case X is necessarily a regular graph of degree $|E(G)|$.The identity map on a fixed set will be denoted by 1. The order of the element x of the group Γ will be denoted by $o(x)$.

Throughout the article we make use of the usual notation: $\Gamma = \mathbb{Z}_n$ for a finite (additive) abelian group, $K_{m,n}$ for the complete bipartite graph with partition sets of sizes m and n, P_n for the path on n vertices, C_n for the cycle with length n, K_n for the complete graph on n vertices, K_1 for an isolated vertex, $G + H$ for the disjoint union $G \cup H$ of G and H, and mG for m disjoint copies of G. Let $n_1, n_2, \ldots, n_r,\ r \geq 1$, be positive integres, $n_1, n_r \geq 1$ and $n_i \geq 0$ for $i \in \{2, 3, \ldots, r-1\}$. The caterpillar C_r $(n_1, n_2, \ldots, n_r)$ is the tree obtained from the path $P_r := x_1x_2 \ldots x_r$ by joining vertex x_i to n_i new vertives, $i \in \{1, 2, \ldots, r\}$.

Let Γ be a finite group and $A \subseteq \Gamma$ a subset of Γ, such that $A^{-1} = A$ and $1 \notin A$. Consider the Cayley graph $X =$Cay$(\Gamma,\ A)$ where $E(X) = \{(x,\ ax) : x \in \Gamma,\ a \in A\}$.To each arc $(x,\ ax)$ of X we assign the colour a. Sometimes a or its inverse will be mentioned as the colours of the corresponding edge. (Note that in [3] and elsewhere it is also assumed that A is a spanning set for Γ. Here however this property is not needed).

The following results were established in [2]

Let Γ be a finite group and σ be a permutation of Γ. We say that σ is balanced if $\sigma(yz)\sigma(xz)^{-1} = \sigma(y)\sigma(x)^{-1}$for all $x,\ y,\ z \in \Gamma$.

Of course, all automorphisms σ of Γ are balanced. Besides, for fixed $a,\ b \in \Gamma$, the map $\sigma(x) = axb$ is a balanced permutation.

Definition 1.1. Let A be any non-empty subset of Γ and let σ be a balanced permutation of Γ. For a map $f : A \to \Gamma$, the map taking $a \in A$ into $f(a^{-1})^{-1}af(a)$ is denoted by f^*. Let us call f a starter map for $(\Gamma,\ A,\ \sigma)$ if f^* is injective and satisfies:

$$yx^{-1} \in A \quad \text{if and only if } \exists\, a \in A \ \ f^*(a)\sigma(x) = \sigma(y). \tag{1.1}$$

For example, if f is a map such that f^* is the restriction to A of some automorphism σ of Γ, then f is a starter map for $(\Gamma,\ A,\ \sigma)$. Namely, $f^*(a)\sigma(x) = \sigma(y)$ if and only if

$\sigma(ax) = \sigma(y)$, which is equivalent to $ax = y$. We also note that if $1 \notin A$ then $f^*(a) \neq 1$ for all $a \in A$, elsewhere from condition (1.1) we would get the contradiction 1= $xx^{-1} \in A$.

Clearly, when using additive notation, a balanced map will be a permutation σ satisfying $\sigma(y+z) - \sigma(x+z) = \sigma(y) - \sigma(x)$. The map f^* will take a into $-f(-a) + a + f(a)$ and condition (1.1) will be replaced by

$$y - x \in A \text{ if and only if } \exists\, a \in A \;\; f^*(a) + \sigma(x) = \sigma(y). \tag{1.2}$$

Let X =Cay$(\Gamma,\ A)$, σ be a balanced permutation of G and f be a starter map for $(\Gamma,\ A,\ \sigma)$. Define $B(f)$ as the collection of graphs

$$\mathcal{P}(x) = \{(f(a)\sigma(x),\ af(a)\sigma(x)) : a \in A\}. \tag{1.3}$$

Theorem 1.1. *Let X =Cay$(\Gamma,\ A)$ and σ be a balanced permutation of Γ. If f is a starter map for $(\Gamma,\ A,\ \sigma)$. Then the collection $B(f)$ forms an ODC of X by $P(1)$. Moreover, the group of right translations $g \mapsto xg$ of Γ form a colour-preserving automorphism group of $B(f)$.*

For more results on ODCs of graphs, see ([1], [4]).

In [5], Sampathkumar et al. completely setteled the existence proplem of CODCs of 4-regular circulant graphs.

In [2], Scapellato et al studied the ODC of Cayley graphs and proved the following.

(i) All 3-regular Cayley graphs, except K_4, have ODCs by P_4.

(ii) All 3-regular Cayley graphs on Abelian groups, except K_4, have ODCs by $P_3 \cup K_2$.

(iii) All 3-regular Cayley graphs on Abelian groups, except K_4 and the 3-prism (Cartesian product of C_3 and K_2), have ODCs by $3K_2$.

In [6], Hartmann and Schumacher proved the following.

(i) Let H be a 2-regular graph. There exists an ODC of H by $2K_2$ with three exceptions for H: $C_3, C_4$and $2C_3$.

(ii) Let H be a 3-regular graph containing a 1-factor and without a component isiomorphic to K_4. There exists an ODC of H by P_4.

(iii) Let H be a 3-regular graph containing a 1-factor and $|V(H)| \geq 24$. There exists an ODC of H by $P_3 + K_2$. The other terminology not defined here can be found in [7].

In this paper, we are almost exclusively concerned with the starter maps of the orthogonal double covers of cayley graphs and using this method to construct ODCs by a complete bipartite graph, a complete tripartite graph, caterpillar, and a connected union of a cycle and a star whose center vertex belongs to that cycle. Note that the number of edges of G is $|A|$, equal to the number of vertices belonging to non-trivial connected components.

2 Results and Discussion

The follwing theorem construct an ODC of Cayley graph by a complete bipartite graph.

Theorem 2.1. *For all positive integers $m, n,\ p$ and $mp = n - 3$, there exists an ODC of $Cay(\mathbb{Z}_n,\ A)$ by $K_{m,p}$.*

Proof. Let $A = \mathbb{Z}_n \backslash \{0, 1, n-1\}$. For each $a \in A$; define the map $f : A \to \mathbb{Z}_n$ by $f(a) = x_0$ if $2 \le a \le p+1$; $f(a) = x_1$ if $p+2 \le a \le 2p+1; \ldots; f(a) = x_{m-1}$ if $(m-1)p+2 \le a \le mp+1$, where $x_j = 1 - jp : 0 \le j \le m-1$. From the defination of $f(a)$, $P(0)$ is isomorphic to the graph $G = K_{m,p}$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in B(f)$. For $jp+2 \le a \le (j+1)p+1$, $f^*(a) = f(a) - f(-a) + a = x_j - x_{m-(j+1)} + a$. And hence f^* is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$.Therefore f^* satisfies condition (1.2) with $\sigma = 1$, which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying theorem (1.1), proves the claim. □

The follwing theorem construct an ODC of Cayley graph by a complete tripartite graph.

Theorem 2.2. *For all positive integers r, s, n and $n = rs$, there exists an ODC of $Cay(\mathbb{Z}_n,\ A) by\ K_{1,(r-1),(s-1)}$.*

Proof. Let $A = \mathbb{Z}_n \backslash \{0\}$. For each $a \in A$; define the map $f : A \to \mathbb{Z}_n$ by $f(a) = 0$ if $1 \le a \le s-1$; $f(a) = (r-1)s$ if $s \le a \le 2s-1$; $f(a) = (r-2)s$ if $2s \le a \le 3s-1$; $\ldots; f(a) = s$ if $(r-1)\,s \le a \le rs-1$. From the defination of $f(a)$; $P(0)$ is isomorphic to the graph $G = K_{1,(r-1),(s-1)}$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in B(f)$. For $1 + ls \le a \le (l+1)s - 1 : 0 \le l \le r-1$; $f^*(a) = f(a) - f(-a) + a = -(2l+1)s + a$; for $a \in \{\ s\ , 2s\ ,\ \ldots,\ (r-1)s\}$; $f^*(a) = f(a) - f(-a) + a = -a$. And hence f^* is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$.Therefore f^* satisfies condition (1.2) with $\sigma = 1$, which implies that $f(a)$is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying theorem (1.1), proves the claim. □

In the following theorem, we construct the orthogonal double covers of Cayley graphs by $C_l \cup^v K_{1,m}$ where m is a positive integer (the union of cycle C_l and a star $K_{1,m}$ whose center vertex v belongs to that cycle).

Theorem 2.3. *Let l, m, n be positive integers such that $l < n$ and $3 \le l \le 9$.Then there exists an ODC of $Cay(\mathbb{Z}_n,\ A)\ by\ C_l \cup^v K_{1,m}$.*

Proof For $3 \le l \le 9$ we define a suitable starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$ in each case of l :

Case $\mathbf{1.}$ $\mathbf{l = 3}$

For $n \ge 5$, $A = \mathbb{Z}_n \backslash \{0\}$ and for each $a \in A$, define $f : A \to \mathbb{Z}_n$ by $f(a) = 0$ if $a = 2$; $f(a) = 4$ if $a \in \{n-2, n-1\}$ and $f(a) = 2$ otherwise. From the defination of $f(a)$; $P(0)$ is isomorphic to the graph $G = C_3 \cup^2 K_{1,n-4}$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in B(f)$. For $a \in \{1, 2, n-2, n-1\}$; $f^*(a) = f(a) - f(-a) + a = -a$; for otherwise, $f^*(a) = f(a) - f(-a) + a = a$. And hence f^* is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$. Therefore f^* satisfies equation condition (1.2) with $\sigma = 1$, which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying theorem (1.1), proves the claim.

Case $\mathbf{2.}$ $\mathbf{l = 4}$

For $n = 2m$, $m > 3$, $A = \mathbb{Z}_{2m} \backslash \{0, m-1, m+1\}$ and for each $a \in A$, define $f : A \to \mathbb{Z}_{2m}$ by $f(a) = m$ if $a \in \{1, 2m-1\}$ and $f(a) = 2m + 1 - a$ otherwise. From the defination of $f(a)$; $P(0)$ is isomorphic to the graph $G = C_4 \cup^1 K_{1,2m-7}$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in B(f)$. For $a \in \{1, m, 2m-1\}$; $f^*(a) = f(a) - f(-a) + a = a$; for otherwise, $f^*(a) = f(a) - f(-a) + a = -a$. And hence f^* is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$. Therefore f^* satisfies condition

(1.2) with $\sigma = 1,$ which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying theorem (1.1), proves the claim.

For $n = 2m+1,\ m \geq 5,\ A = \mathbb{Z}_{2m+1}\backslash\{0, m-1, m+2\}$ and for each $a \in A,$ define $f : A \to \mathbb{Z}_{2m+1}$ by $f(a) = 0$ if $a = 1;\ f(a) = 2$ if $a = 2m$ and $f(a) = m+3$ otherwise. From the defination of $f(a);\ P(0)$ is isomorphic to the graph $G = C_4 \cup^{m+3} K_{1,2m-6}$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in B(f)$. For $a \in \{1, 2m\};\ f^*(a) = f(a) - f(-a) + a = -a;$ for otherwise, $f^*(a) = f(a) - f(-a) + a = a$. And hence f^* is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$. Therefore f^* satisfies condition (1.2) with $\sigma = 1,$ which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying theorem (1.1), proves the claim.

Case $\mathbf{3}$. $\mathbf{l = 5}$

For $n > 10,\ A = \mathbb{Z}_n\backslash\{0, 4, n-4\}$ and for each $a \in A,$ define $f : A \to \mathbb{Z}_n$ by $f(a) = 2$ if $a \in \{2, n-1\};\ f(a) = 6$ if $a = n-2$ and $f(a) = 0$ otherwise. From the defination of $f(a);\ P(0)$ is isomorphic to the graph $G = C_5 \cup^0 K_{1,n-8}$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in B(f)$. For $a \in \{1, 2, n-2, n-1\};\ f^*(a) = f(a) - f(-a) + a = -a;$for otherwise, $f^*(a) = f(a) - f(-a) + a = a$. And hence f^* is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$. Therefore f^* satisfies condition (1.2) with $\sigma = 1,$ which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying theorem (1.1), proves the claim

Case $\mathbf{4}$. $\mathbf{l = 6}$

For $n > 7,\ A = \mathbb{Z}_n\backslash\{0\}$ and for each $a \in A,$ define $f : A \to \mathbb{Z}_n$ by $f(a) = 4$ if $a \in \{1, n-2\};\ f(a) = 0$ if $a \in \{2, 3\};\ f(a) = 6$ if $a \in \{n-3, n-1\}$and $f(a) = 3$ otherwise. From the defination of $f(a),\ P(0)$ is isomorphic to the graph $G = C_6 \cup^3 K_{1,n-7}$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in B(f)$. For $a \in \{1, 2, 3, n-3, n-2, n-1\};\ f^*(a) = f(a) - f(-a) + a = -a;$ for otherwise, $f^*(a) = f(a) - f(-a) + a = a$. And hence f^* is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$. Therefore f^* satisfies condition (1.2) with $\sigma = 1,$ which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying theorem (1.1), proves the claim.

Case $\mathbf{5}$. $\mathbf{l = 7}$

For $n = 2m+1, m > 5, A = \mathbb{Z}_{2m+1}\backslash\{0, 4, 6, 2m-5, 2m-3\}$ and for each $a \in A,$ define $f : A \to \mathbb{Z}_{2m+1}$ by $f(a) = 1$ if $a \in \{1, 2m\};\ f(a) = 2$ if $a = 2;\ f(a) = 6$ if $a = 2m-1;\ f(a) = 2m-1$ if $a \in \{8, 2m-7\}$ and $f(a) = 0$ otherwise. From the defination of $f(a); P(0)$ is isomorphic to the graph $G = C_7 \cup^0 K_{1,2m-11}$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in B(f)$. For $a \in \{2, 2m-1\};\ f^*(a) = f(a) - f(-a) + a = -a;$ for otherwise, $f^*(a) = f(a) - f(-a) + a = a$. And hence f^* is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$. Therefore f^* satisfies condition (1.2) with $\sigma = 1,$ which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying theorem (1.1), proves the claim.

Case $\mathbf{6}$. $\mathbf{l = 8}$

For $n \geq 14,\ A = \mathbb{Z}_n\backslash\{0, 2, 4, n-2, n-4\}$ and for each $a \in A,$ define $f : A \to \mathbb{Z}_n$ by $f(a) = 1$ if $a \in \{1, n-1\}; f(a) = 5$ if $a \in \{3, n-3\};\ f(a) = 3$ if $a \in \{5, n-5\}$and $f(a) = n+6-a$ otherwise. From the defination of $f(a);\ P(0)$ is isomorphic to the graph $G = C_8 \cup^6 K_{1,n-13}$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in B(f)$. For $a \in \{1, 3, 5, n-5, n-3, n-1\};$ $f^*(a) = f(a) - f(-a) + a = a;$ for otherwise, $f^*(a) = f(a) - f(-a) + a = -a$. And hence f^* is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$. Therefore f^* satisfies condition (1.2) with $\sigma = 1,$ which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying theorem (1.1), proves the claim.

Case $\mathbf{7}$. $\mathbf{l = 9}$

For $n = 2m+1, m > 6, A = \mathbb{Z}_{2m+1}\backslash\{0, 4, 8, 2m-7, 2m-3\}$ and for each $a \in A,$ define $f : A \to \mathbb{Z}_{2m+1}$ by $f(a) = 1$ if $a \in \{1, 2m\};\ f(a) = 2m-3$ if $a \in \{2, 2m-1\};\ f(a) = 2$ if $a = 3;\ f(a) = 3$ if $a \in \{5, 2m-4\};\ f(a) = 8$ if $a = 2m-2$ and $f(a) = 0$ otherwise. From the defination of $f(a); P(0)$ is isomorphic to the graph $G = C_9 \cup^0 K_{1,2m-13}$ has edges $E(G) = \{(f(a),\ f(a)+$

$a) : a \in A\} \in B(f)$. For $a \in \{3, 2m-2\}$; $f^*(a) = f(a) - f(-a) + a = -a$; for otherwise, $f^*(a) = f(a) - f(-a) + a = a$. And hence f^* is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$. Therefore f^* satisfies condition (1.2) with $\sigma = 1$, which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying theorem (1.1), proves the claim. □

In the following theorem, we construct the orthogonal double covers of Cayley graphs by a caterpillar .

Theorem 2.4. *Let r, n, p, q be positive integers such that $2 \le r \le 6,\ r < n$. Then there exists an ODC of $Cay(\mathbb{Z}_n,\ A)$ by $C_r(p, 0, 0, \ldots 0, q)$.*

Proof. For $2 \le r \le 6$ we define a suitable starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$ in each case of r :

Case $\mathbf{1.}$ $\mathbf{r = 2}$

For $n \ge 8$, $A = \mathbb{Z}_n \backslash \{0, 3, n-3\}$ and for each $a \in A$, define $f : A \to \mathbb{Z}_n$ by $f(a) = 0$ if $a = 1$; $f(a) = 1$ if $a \in \{2, n-2\}$; $f(a) = 2$ if $a = n-1$ and $f(a) = n+2-a$ otherwise. From the defination of $f(a)$; $P(0)$ is isomorphic to the graph $G = C_2(3, n-7)$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in B(f)$. For $a \in \{2, n-2\}$, $f^*(a) = f(a) - f(-a) + a = a$; for otherwise $f^*(a) = f(a) - f(-a) + a = -a$. And hence f^* is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$. Therefore f^* satisfies condition (1.2) with $\sigma = 1$, which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying theorem (1.1), proves the claim.

Case $\mathbf{2.}$ $\mathbf{r = 3}$

For $n = 2m$, $m \ge 5$, $A = \mathbb{Z}_{2m} \backslash \{0, 2, 3, 2m-3, 2m-2\}$ and for each $a \in A$, define $f : A \to \mathbb{Z}_{2m}$ by $f(a) = m+2$ if $a \in \{1, m-1, 2m-1\}$ and $f(a) = m$ otherwise. From the defination of $f(a)$; $P(0)$ is isomorphic to the graph $G = C_3(2, 0, 2m-9)$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in B(f)$. For $a \in \{m-1, m+1\}$; $f^*(a) = f(a) - f(-a) + a = -a$; for otherwise, $f^*(a) = f(a) - f(-a) + a = a$. And hence f^* is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$. Therefore f^* satisfies condition (1.2) with $\sigma = 1$, which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying theorem (1.1), proves the claim.

For $n = 2m+1$, $m \ge 5$, $A = \mathbb{Z}_{2m+1} \backslash \{0, 1, 3, 2m-2, 2m\}$ and for each $a \in A$, define $f : A \to \mathbb{Z}_{2m+1}$ by $f(a) = m+2$ if $a \in \{2, m, 2m-1\}$ and $f(a) = m+1$ otherwise. From the defination of $f(a)$; $P(0)$ is isomorphic to the graph $G = C_3(2, 0, 2m-8)$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in B(f)$. For $a \in \{m, m+1\}$, $f^*(a) = f(a) - f(-a) + a = -a$; for otherwise, $f^*(a) = f(a) - f(-a) + a = a$. And hence f^* is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$. Therefore f^* satisfies condition (1.2) with $\sigma = 1$, which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying theorem (1.1), proves the claim.

Case $\mathbf{3.}$ $\mathbf{r = 4}$

For $n = 2m$, $m > 3$, $A = \mathbb{Z}_{2m} \backslash \{0, m,\}$ and for each $a \in A$, define $f : A \to \mathbb{Z}_{2m}$ by $f(a) = 0$ if $a = 1$; $f(a) = 2$ if $a = 2m-1$; $f(a) = 1$ if $a = m-1$; $f(a) = 2m-1$ if $a = m+1$ and $f(a) = m+1$ otherwise. From the defination of $f(a)$; $P(0)$ is isomorphic to the graph $G = C_4(2, 0, 0, 2m-7)$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in B(f)$. For $a \in \{1, m-1, m+1, 2m-1\}$; $f^*(a) = f(a) - f(-a) + a = -a$; for otherwise, $f^*(a) = f(a) - f(-a) + a = a$. And hence f^* is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$. Therefore f^* satisfies condition (1.2) with $\sigma = 1$, which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying theorem (1.1), proves the claim.

For $n = 2m+1$, $m > 3$, $A = \mathbb{Z}_{2m+1} \backslash \{0, m, m+1\}$ and for each $a \in A$, define $f : A \to \mathbb{Z}_{2m+1}$ by $f(a) = 0$ if $a = 1$; $f(a) = 2$ if $a = 2m$; $f(a) = 1$ if $a = m-1$; $f(a) = 2m-1$ if $a = m+2$

and $f(a) = m+1$ otherwise. From the defination of $f(a)$; $P(0)$ is isomorphic to the graph $G = C_4(2,0,0,2m-7)$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in B(f)$. For $a \in \{1, m-1, m+2, 2m\}$, $f^*(a) = f(a)-f(-a)+a = -a$; for otherwise, $f^*(a) = f(a)-f(-a)+a = a$. And hence f^* is injective as well as surjective because of $\{f(a)-f(-a)+a : a \in A\} = A$. Therefore f^* satisfies condition (1.2) with $\sigma = 1$, which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying theorem (1.1), proves the claim.

Case **4.** $\mathbf{r = 5}$

For $n = 2m$, $m > 4$, $A = \mathbb{Z}_{2m} \backslash \{0, m-2, m+2\}$ and for each $a \in A$, define $f : A \to \mathbb{Z}_{2m}$ by $f(a) = m$ if $a \in \{2, 2m-2\}$; $f(a) = 2$ if $a \in \{m, 2m-1\}$ and $f(a) = 0$ otherwise. From the defination of $f(a)$; $P(0)$ is isomorphic to the graph $G = C_5(1,0,0,0,2m-8)$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in B(f)$. For $a \in \{1, 2m-1\}$; $f^*(a) = f(a) - f(-a) + a = -a$; for otherwise, $f^*(a) = f(a) - f(-a) + a = a$. And hence f^* is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$. Therefore f^* satisfies condition (1.2) with $\sigma = 1$, which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying theorem (1.1), proves the claim.

For $n = 2m+1$, $m > 3$, $A = \mathbb{Z}_{2m+1} \backslash \{0, 1, 2m\}$ and for each $a \in A$, define $f : A \to \mathbb{Z}_{2m+1}$ by $f(a) = 1$ if $a \in \{m-1, m+1\}$; $f(a) = 2$ if $a = m$; $f(a) = 2m-1$ if $a = m+2$ and $f(a) = m+1$ otherwise. From the defination of $f(a)$; $P(0)$ is isomorphic to the graph $G = C_5(1,0,0,0,2m-7)$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in B(f)$. For $a \in \{m-1, m, m+1, m+2\}$; $f^*(a) = f(a)-f(-a)+a = -a$; for otherwise $f^*(a) = f(a)-f(-a)+a = a$. And hence f^* is injective as well as surjective because of $\{f(a)-f(-a)+a : a \in A\} = A$. Therefore f^* satisfies condition (1.2) with $\sigma = 1$, which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying theorem (1.1), proves the claim.

Case **5.** $\mathbf{r = 6}$

For $n > 10$, $A = \mathbb{Z}_n \backslash \{0, 4, n-4\}$ and for each $a \in A$, define $f : A \to \mathbb{Z}_n$ by $f(a) = 5$ if $a \in \{1, n-1\}$; $f(a) = 0$ if $a \in \{2, n-2\}$; $f(a) = 1$ if $a \in \{3, n-3\}$ and $f(a) = n+2-a$ otherwise. From the defination of $f(a)$; $P(0)$ is isomorphic to the graph $G = C_6(1,0,0,0,0,n-9)$ has edges $E(G) = \{(f(a),\ f(a)+a) : a \in A\} \in B(f)$. For $a \in \{1,2,3,n-3,n-2,n-1\}$; $f^*(a) = f(a) - f(-a) + a = a$; for otherwise, $f^*(a) = f(a) - f(-a) + a = -a$. And hence f^* is injective as well as surjective because of $\{f(a) - f(-a) + a : a \in A\} = A$. Therefore f^* satisfies condition (1.2) with $\sigma = 1$, which implies that $f(a)$ is a starter map with respect to $(\mathbb{Z}_n,\ A,\ 1)$. Applying theorem (1.1), proves the claim. □

3 Conclusion

In conclusion, we pose the following conjectures:

Conjecture 1. we conjecture that if l, m, n are positive integers where $n > l$ and $n > m$ there is an ODC of a Cayley graph $Cay(\mathbb{Z}_n,\ A)$ where $|A| = m$ by $C_l \cup^v K_{1,m-l}$.

Conjecture 2. we conjecture that if m, n, r, i are positive integers where $n > r$, $n > m$ and $n > i$ there is an ODC of a Cayley graph $Cay(\mathbb{Z}_n,\ A)$ where $|A| = m$ by $C_r(i, 0, 0, \ldots 0, m-(i+r-1))$.

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Competing Interests

The authors declare that no competing interests exist.

References

[1] Gronau HDOF, Grüttmüller M, Hartmann S, Leck U, Leck V. On orthogonal double covers of graphs, Des. Codes Cryptogr. 2002;27:49-91.

[2] Scapellato R, El Shanawany R, Higazy M. Orthogonal double covers of Cayley graphs, Discrete Appl. Math. 2009;157:3111-3118.

[3] Lauri J, Scapellato R. Topics in graph automorphisms and reconstruction, London Math. Soc. S.T. 54, Cambridge University Press, 2003.

[4] El Shanawany R, Higazy M, and El Mesady A. On Cartesian Products of Orthogonal Double Covers, International Journal of Mathematics and Mathematical Sciences. 2013; 265136:1-4.

[5] Sampathkumar R, Srinivasan S. Cyclic orthogonal double covers of 4-regular circulant graphs, Discrete Mathematics. 2011;311:2417-2422.

[6] Hartmann S, Schumacher U. Orthogonal double covers of general graphs, Discrete Appl. Math. 2004; 138:107-116.

[7] Balakrishnan R, Ranganathan K. A Textbook of Graph Theory. Springer. Berlin, 2012.

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