Journal of Scientific Research & Reports
3(21): 2792-2800, 2014; Article no. JSRR.2014.21.007
ISSN: 2320–0227

SCIENCEDOMAIN ***international***
www.sciencedomain.org

Direct Solution of Initial Value Problems of Fourth Order Ordinary Differential Equations Using Modified Implicit Hybrid Block Method

S. J. Kayode[1], M. K. Duromola[1] and Bolarinwa Bolaji[2*]

[1]Department of Mathematical Sciences, Federal University of Technology, Akure, Nigeria.
[2]Department of Computer Sciences, Salem University, Lokoja, Nigeria.

Authors' contributions

This work was carried out in cooperation between all authors. Author SJK designed the study, wrote the concept of the article, taken care of formatting and provided finishing touch to this manuscript. Author MKD managed the experimental process and wrote the first draft of the manuscript. Author BB too managed the experimental process, along with the literature searches and analyses of the study. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JSRR/2014/11953
Editor(s):
(1) Narcisa C. Apreutesei, Technical University of Iasi, Romania.
Reviewers:
(1) Hesam-Eldien Derili Gherjalar, Department of Mathematics, Karaj branch, Islamic Azad University, Karaj, Iran.
(2) Anonymous, Manipal University, India.
(3) Anonymous, Baha-Ud-Din University, Multan, Pakistan.
Peer review History: http://www.sciencedomain.org/review-history.php?iid=632&id=22&aid=5926

Original Research Article

Received 11th June 2014
Accepted 24th July 2014
Published 1st September 2014

ABSTRACT

Our focus in this article is the derivation; analysis and implementation of a new modified implicit hybrid block method for the direct solution of initial value problems of fourth order ordinary differential equations. In the derivation of the method, we adopted the approach of collocation approximation to obtain the main scheme with continuous coefficients. From the main scheme, additional schemes were developed. The implementation strategy of the new method is by combining the main scheme and the additional schemes as

**Corresponding author: E-mail: bolarinwa.bolaji@yahoo.com;*

simultaneous integrator to initial value problem of fourth order ordinary differential equations. As required of any numerical method, the properties analysis of the block was done and the result showed that it is consistent, convergent, zero stable and absolutely stable. We then test our method with numerical examples solved using existing method and were found to give better results.

Keywords: Interpolation; continuous coefficients; block method; numerical integration; fourth order ordinary differential equations.

1. INTRODUCTION

Some empirical problems and physical. Phenomena in science and engineering, such as mechanical systems without dissipation, celestial mechanics, control theory, computer aided designs when modeled result to higher order ordinary differential equations of the form:

$$\begin{aligned} &y^{(iv)} = f(x, y, y', y'', y'''), y(x_0) = \eta_0, \\ &y'(x_0) = \eta_1, y''(x_0) = \eta_2, y'''(x_0) = \eta_3 \end{aligned} \tag{1}$$

Conventionally, to solve (1) numerically, we first reduce it to system of first order ordinary differential equations and then apply any other existing first order method to solve it. Many literature abounds on this [1,2]. The drawback of this method is that it is time consuming, cumbersome to solve, and take much computer space. To circumvent these draw backs, many researchers have solved (1) directly, they include: [3,4,5,6] who developed block methods for numerical solution of fourth order ordinary differential equations. The works of [5,6] serve as improvement on the work of [4] who developed Linear multistep method for the solution of fourth order ordinary differential equations whose implementation is Predictor – Corrector mode.

We are motivated to advance the course of research work by continuing with the proposition of block method which have been shown to eliminate the drawbacks of Predictor corrector method as discussed in [5,7,8,9,10,11] in their works have proposed single Step hybrid methods for the direct numerical solution of initial value problems of second order and third order Ordinary differential equations respectively. In all Cases, their methods of implementation are block mode with the proposed methods being efficient, adequate and suitable towards catering for the class of problems for which they were designed.

Consequently, our motivation in this work is the success story of the adoption of single step method to solving higher order ordinary differential equations. Thus, in this work, we are proposing a single step method for the direct numerical solution of fourth order ordinary differential equations, which eliminates the use of predictors by providing sufficiently accurate simultaneous difference equations from a single continuous formula and its derivatives.

According to [12], the general block formula is given by:

$$Y_m = ey_n + h^{\mu} df(y_n) + h^{\mu} bF(y_m) \tag{2}$$

Where e is $s \times s$ vector, d is r - vector and b is $r \times r$ vector, s is the interpolation points and r is the *collection points. F is a k – vector whose* J^{th} entry is $f_{n+j} = f(t_{n+j,} y_{n+j})$, μ is the order of the differential equation.

Given a predictor equation in the form:

$$Y_m^{(0)} = ey_n + h^{\mu} df(y_n). \tag{3}$$

By Putting (3) in (2) we have:

$$Y_m = ey_n + h^{\mu} df(y_n) + h^{\mu} bF(ey_n + h^{\mu} dfy_n). \tag{4}$$

Equation (4) is called a self starting block-predictor-corrector method because the prediction equation is gotten directly from the block formula [13,4].

Consequently, our focus in this paper is the proposition of an improved implicit continuous hybrid algorithm for the solution of initial value problems of fourth order ordinary differential equations.

2. DERIVATION OF THE METHODS

We take our basis function to be a power series of the form:

$$y(x) = \sum_{j=0}^{r+s} a_j x^j \tag{5}$$

The third derivative of (5) gives:

$$y^{(iv)}(x) = \sum_{j=0}^{r+s} j(j-1)(j-2)(j-3) a_j x^{j-4} \tag{6}$$

By putting (6) into (1) we have the differential system:

$$\sum_{j=0}^{r+s} j(j-1)(j-2)(j-3) a_j x^{j-4} = f(x, y(x), y'(x), y''(x), y'''(x)) \tag{7}$$

Where a_j the parameters to be determined are, while r+s denotes the number of collocation and interpolation points. By collocating (7) at the mesh points $x = x_{n+j}$, $j = 0(\frac{1}{5})1$, and interpolating (5) at $x = x_{n+j}$, $j = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ yields a system of equations:

$$\sum_{j=0}^{r+s} a_j x^j = y_{n+s} \tag{8}$$

$$\sum_{j=0}^{r+s} j(j-1)(j-2)(j-3)a_j x^{j-4} = f_{n+r} \tag{9}$$

By putting these system of equations in matrix form and then solved to obtain the values of Parameters a_j's , j = 0, $1/5$, . Which when substituted in (5), yields, after some manipulation, a hybrid linear method with continuous coefficients of the form:

$$y(x) = \sum_{j=0}^{1} \alpha_j y_{n+j}(x) + h^4 \sum_{j=0}^{1} \beta_j f_{n+j}(x) \tag{10}$$

The co efficient of $\alpha_j(x)$ and β_j are:

$\alpha_0(t) = 1/3\left(t^3 + t^2 + t\right)$ $\quad\quad$ $\alpha_{1/5}(t) = -1/5\left(t^3 + 15t^2 + 4t\right)$

$\alpha_{2/5}(t) = 1/3\left(t^3 + 16t^2 + t\right)$ $\quad\quad$ $\alpha_{3/5}(t) = -1/5\left(t^3 + 2t^2 + 15t + 2\right)$

$\alpha_{4/5}(t) = 1/7\left(t^3 + t^2 + 3t + 2\right)$

$$\beta_0(t) = \frac{1}{450{,}000}(t^8 + 4t^7 - t^6 + 2t^5 - 3t^4 + t^3 - 4t - 1)$$

$$\beta_{1/5}(t) = \frac{1}{450{,}000}(3t^8 - 5t^7 - 124t^6 + 9t^5 - 3t^4 - 9t^3 + 5t)$$

$$\beta_{2/5}(t) = \frac{1}{450{,}000}(t^8 - 6t^7 - 2t^6 - 200t^5 - 274t^4 + 2t^3 - t - 6)$$

$$\beta_{3/5}(t) = \frac{1}{450{,}000}(4t^8 - 3t^7 + t^6 - 120t^5 - 4t^4 - t^3 + 3t - 4)$$

$$\beta_{4/5}(t) = \frac{1}{450{,}000}(t^8 + 9t^7 - 2t^6 + 7t^5 - 6t^4 - t^3 + 2t - 9) \tag{11}$$

Where $t = \dfrac{x - x_n}{h}$

2.1 Derivation of the Block

The general block formula proposed by Awoyemi et al. [12], in the Normalized form is given by:

$$A^{(0)}Y_m = ey_n + h^{\mu-\lambda}df\left(y_n\right) + h^{\mu-\lambda}bF\left(y_m\right) \tag{12}$$

By evaluating (10) at $t=1$; the first, second and the third derivative at $x = x_{n+1}, i = 0(\frac{1}{5})1$ and substituting into (12) gives its coefficients as:

$$d = \begin{bmatrix} \frac{49126}{1134000000} & \frac{4264}{8859375} & \frac{25488}{14000000} & \frac{40448}{450000} & \frac{3346}{362880} & \frac{3229}{5040000} & \frac{317}{78750} & \frac{5481}{560000} & \frac{712}{39375} & \frac{233}{8064} \\ \frac{2462}{252000} & \frac{5680}{252000} & \frac{984}{28000} & \frac{376}{7875} & \frac{122}{2016} & \frac{475}{7200} & \frac{4704}{75600} & \frac{51}{800} & \frac{14}{255} & \frac{19}{288} \end{bmatrix}^T$$

$$e = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{50} & \frac{2}{25} & \frac{9}{50} & \frac{8}{25} & \frac{1}{2} & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{750} & \frac{4}{375} & \frac{9}{250} & \frac{32}{375} & \frac{1}{6} & \frac{1}{50} & \frac{2}{25} & \frac{9}{50} & \frac{8}{25} & \frac{1}{2} & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$$

$A^0 = 20 \times 20$ identity matrix

$$B = \left[\begin{matrix} \frac{2780}{30965760} & \frac{15104}{7741440} & \frac{34668}{3440640} & \frac{114688}{3870720} & \frac{2140}{1290240} & \frac{21248}{1290240} & \frac{67068}{1290240} & \frac{139264}{1290240} \\ \frac{-8472}{30965760} & \frac{-10107}{7741440} & \frac{-11492}{3440640} & \frac{-15675}{3870720} & \frac{-2365}{1290240} & \frac{-8809}{1290240} & \frac{-9877}{1290240} & \frac{5015}{1290240} \\ \frac{716}{30965760} & \frac{2816}{7741440} & \frac{4860}{3440640} & \frac{16384}{3870720} & \frac{516}{1290240} & \frac{3328}{1290240} & \frac{8100}{1290240} & \frac{24576}{1290240} \\ \frac{-131}{30965760} & \frac{-512}{7741440} & \frac{-891}{3440640} & \frac{-2560}{3870720} & \frac{-94}{1290240} & \frac{-608}{1290240} & \frac{1458}{1290240} & \frac{-2560}{1290240} \end{matrix}\right.$$

$$\left.\begin{matrix} \frac{45900}{1958400} & \frac{32256}{322560} & \frac{156}{2560} & \frac{24}{90} & \frac{-264}{2880} & \frac{2356}{6840} & \frac{102}{320} & \frac{32}{90} \\ \frac{-23970}{1958400} & \frac{-6735}{322560} & \frac{18}{2560} & \frac{6}{90} & \frac{-264}{2880} & \frac{456}{6840} & \frac{72}{320} & \frac{12}{90} \\ \frac{9860}{1958400} & \frac{3584}{322560} & \frac{20}{2560} & \frac{8}{90} & \frac{106}{2880} & \frac{76}{6840} & \frac{42}{320} & \frac{32}{90} \\ \frac{-1792}{1958400} & \frac{-672}{322560} & \frac{-3}{2560} & 0 & \frac{-19}{2880} & \frac{-18}{6840} & \frac{-3}{320} & \frac{7}{90} \end{matrix}\right]^T$$

3. ANALYSIS OF THE PROPERTIES OF THE BLOCK

In this section we carry out the analysis of the Basic properties of the new method.

3.1 Order of the Method

The linear operator of the block (12) is defined as:

$$L\{y(x):h\} = Y_m - ey_m + h^{\mu-\lambda} df(y_m) + h^{\mu-\lambda} bF(y_m) \tag{13}$$

By expanding $y(x_n + ih)$ and $f(x_n + jh)$ in Taylor series, (12) becomes:

$$\begin{aligned} L\{y(x):h\} = C_0 y(x) + C_1 hy'(x) + C_2 h^2 y''(x) + \ldots \\ + C_p h^p y^{(p)}(x) + \end{aligned} \tag{14}$$

The block (11) and associated linear operator are said to have order p if

$$C_0 = C_1 = \ldots = C_{p+1} = 0, C_{p+2} \neq 0.$$

The term C_{p+2} is called the error constant and implies that the local truncation error is given by:

$$t_{n+k} = C_{p+2} h^{(p+2)} y^{(p+2)}(x_n) + 0h^{(p+3)} \tag{15}$$

Hence the block (12) has order 8 with error constant:

$$C_{p+2} = \begin{bmatrix} \frac{1355}{6193152}, \frac{1129}{2580480}, \frac{1129}{1720320}, \frac{71131}{81283860}, \\ \frac{6117}{671420}, \frac{2417}{321560}, \frac{5227}{1638200}, \frac{6226}{31180194}, \\ \frac{2104}{891019}, \frac{1835}{287923}, \frac{1292}{298142}, \frac{2155}{893287}, \\ \frac{5105}{1935152}, \frac{4221}{2819456}, \frac{1942}{1834231}, \frac{28182}{341940} \end{bmatrix}^T$$

3.2 Zero Stability of the Block

The block (11) is said to be Zero stable if the roots $z_s = 1,2,\ldots,N$ of the characteristic polynomial $\rho(z) = \det(zA - E)$, satisfies $|z| \leq 1$ and the root $|z| = 1$ has multiplicity not exceeding the order of the differential equation. Moreover as $h^{\mu} \to 0, \rho(z) = z^{r-\mu}(\lambda - 1)$, Where μ is the order of the differential equation, for the block (11), $r = 16, \mu = 4$

$$\rho(z) = \lambda^{12}(\lambda - 1)^4 = 0$$
$$\Rightarrow \lambda = 0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1$$

Hence our method is Zero stable.

3.3 Convergence

The necessary and sufficient condition for a numerical method to be convergent is for it to be Zero stable and has order $p \geq 1$, Since our method has been shown to be zero stable and has order 8, it satisfied the above condition, thus our method is convergent.

4. NUMERICAL EXPERIMENTS

To test the accuracy, workability and suitability of the method, we adopted our method to solving some initial value problems of fourth order ordinary differential equations.

Test problem 1.

We consider a non linear fourth order problem:

$$y^{(iv)} = (y')^2 - y(y'') - 4x^2 + e(1 - 4x + x^2) \qquad 0 \leq x \leq 1, y(0) = 0, y'(0) = 1, y''(0) = 3, y'''(0) = 1,$$
$$h = 1/32$$

Whose exact solution is given by: $y(x) = x^2 + e^x$. The result is as shown in Table 1.

Table 1. Showing results for problem 1

XVAL	ERC	NRC	ERR
0.103125	1.119264744787591900	1.119264744787634518	4.261834E-14
0.206250	1.271599493198048300	1.271599493198779782	7.314820E-13
0.306250	1.452110907065012200	1.452110907066637521	1.625321E-12
0.406250	1.666216862500120800	1.666216862506568972	6.568972E -12
0.506250	1.915347109920913400	1.915347109923762547	2.849147E -12
0.603125	2.201081767908965600	2.201081767913196057	4.231457E -12
0.703125	2.514440293337009000	2.514440293349477520	1.246852E -11
0.803125	2.877516387746618300	2.877516387798079728	5.146142E - 11
0.903125	3.282936158805117400	3.282936158837654230	3.765423E - 11
1.003125	3.733049511495201100	3.733049511556738124	7.218014E - 11

Test problem 2.

We consider special fourth order problem:

$$y^{iv} = x; \qquad y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 0, h = 0.1$$

Whose exact solution is: $y(x) = \dfrac{x^5}{120} + x$

Our method was used to solve the problem and result compared with [14]. The result is as shown in Table 2.

Table 2. Showing results for problem 2

XVAL	ERC	NRC	ERR	ERR in [14]
0.1	0.10000008333334000	0.10000008333351720	1.832E-13	7.000E- 10
0.2	0.20000266666666900	0.20000266667150250	4.835E-12	8.999E -10
0.3	0.30002025000000004	0.30002025000721480	7.214E -12	2.999E- 09
0.4	0.40000853333333333	0.40000853340160457	6.832E -11	5.100E- 09
0.5	0.50026041666666665	0.50026041674083458	7.416E -11	7.799E- 09
0.6	0.60064800000000007	0.60064800002714565	2.714E -11	1.180E -08
0.7	0.70140058333333344	0.70140058361478378	2.815E -10	1.240E- 08
0.8	0.80273066666666670	0.80273066700848838	3.412E -10	1.410E -08
0.9	0.90492075000000005	0.90492075019356814	1.936E -10	1.880E- 08
1.0	1.00833333333333000	1.00833333361984509	2.865E - 10	2.600E -08

4.1 Numerical Results

We make use of the following Notations in the table of results:

XVAL: Value of the independent variable where numerical value is taken.
ERC: Exact result at XVAL
NRC: Our Numerical result at XVAL
ERR: Error of our result at XVAL.

5. CONCLUSION

In this paper, we have proposed a modified Implicit Hybrid Block algorithm for the numerical solution of initial value problems of fourth order ordinary differential equations. For better performance of the method, step size is chosen within the stability interval. The results of our new method when compared with the block method proposed by [14] showed that our method is more accurate.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

REFERENCES

1. Lambert JD. Computational methods in ordinary differential equations. John Wiley and Sons Inc, New York; 1973.
2. Fatunla SO. Numerical methods for initial value problems in ordinary differential equation. Academic Press Inc, Harcourt Brace Jovanovich publisher; 1988.

3. Omar Z. Developing parallel block method for solving higher orders ODES directly. PhD Thesis, University putra, Malaysia; 1999.
4. Kayode SJ. A zero stable method for direct solution of fourth order ordinary differential equation. American Journal of Applied Sciences. 2008;5(11):1461-1466.
5. Olabode BT. A six-step scheme for the solution of fourth order ordinary differential equations. Pacific Journal of Science and Technology. 2009;10(1);143-148.
6. Adesanya AO. Block methods for the solutions of general higher order initial value problems of ordinary differential equations. A Ph.D thesis in Mathematical Sciences department of Federal University of Technology, Akure, Nigeria; 2011.
7. Yusuph Y. Some theories and application of linear multistep methods for Ordinary differential equations. Ph.D Thesis University of Jos, Nigeria; 2004.
8. Anake. Continuous Implicit hybrid one-step methods for solutions of initial value problems of general second order ordinary differential equations. Ph.D thesis, Convenant University; 2011.
9. Bolarinwa Bolaji. Implicit hybrid block methods for the numerical solutions of initial value promlems of third order ordinary differential equations. A Ph.D Thesis in Mathematical Sciences Department of Federal University of Technology, Akure; Nigeria; 2012.
10. Bolarinwa Bolaji, Ademiluyi RA, Awoyemi DO, Ogundele JO. A single step implicit hybrid numerical method for the numerical integration of initial value problems of third order ordinary differential equations. Canadian Journal on Science and Engineering Mathematics. 2012;3-4.
11. Bolarinwa Bolaji, Ademiluyi RA, Olaseni Tunde, Duromola MK. A new implicit hybrid block method for the direct solution initial value problems of general third order ordinary differential equations. Canadian Journal on Science and Engineering Mathematics. 2012;2:3.
12. Awoyemi DO, Adebile EA, Adesanya AO, Anake TA. Modified block method for direct solution of second order ordinary differential equation. International Journal of Applied Mathematics and Computation. 2011;3(3):181-188.
13. Shampine LF, Watts HA. Block implicit one step methods. Journal of Math of Computation. 1969;23(108):731-740.
14. Mohammed U. A six step block method for solution of fourth order ordinary differential equations. The Pacific Journal of Science and Technology. 2010;11(1):258-265.

Peer-review history:
The peer review history for this paper can be accessed here:
http://www.sciencedomain.org/review-history.php?iid=632&id=22&aid=5926