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On the Entire Solutions of a Nonlinear Differential Equation of Hayman

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Abstract

Aims/ objectives: Hayman [1] proposed to study the meromorphic solutions of nonlinear differential equations of the form:

$$ff'' - (f')^2 = k_0 + k_1 f + k_2 f' + k_3 f'',$$

where $k_j\ (j = 0, 1, 2, 3)$ are constants. In this note, by using a new method, we give a unified and simplified proof for these known results.

Keywords: Entire solutions; differential equation; Nevanlinna theory

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1 Introduction and Main Results

In this paper, we shall adopt the standard notations in Nevanlinna's value distribution theory of meromorphic functions. For example, the characteristic function $T(r, f)$, the counting function of the poles $N(r, f)$, and the proximity function $m(r, f)$ (see, e.g., [2], [3] and [4]).

The behavior of meromorphic solutions of differential equations has been the subject of much study. Research has concentrated on the value distribution of meromorphic solutions and their rates

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of growth. The purpose of the present paper is to show that a thorough search will yield a list of all meromorphic solutions of a multi-parameter ordinary differential equation introduced by Hayman. Hayman [1] proposed to study the meromorphic solutions of nonlinear differential equations of the form:

$$ff'' - (f')^2 = k_0 + k_1 f + k_2 f' + k_3 f'', \tag{1.1}$$

where k_j $(j = 0, 1, 2, 3)$ are constants. By letting $\omega = f - k_3$, the differential equation (1.1) can be rewritten as

$$\omega\omega'' - (\omega')^2 = \alpha\omega + \beta\omega' + \gamma, \tag{1.2}$$

where α, β, γ are constants.

The major result concerning the order of growth of meromorphic solutions of first-order differential equations is the following theorem due to Gol'dberg [5]. A generalization of Gol'dberg's result to second-order algebraic equations has been conjectured by Bank [6]. Steinmetz [7] proved related results for any second-order polynomial equation which is homogeneous in its dependent variable and its derivatives. Chiang and Halburd [8] studied the Hayman's equation, and they obtain the following results.

Theorem A If not both α and γ are zeros and $\beta \neq 0$, then the meromorphic solutions of (1.2) are

$$\omega(z) = c_1 \exp(\frac{\alpha z}{a_{\mp}}) - \frac{\gamma}{\alpha}, \text{ if } \alpha \neq 0,$$

and

$$\omega(z) = c_1 + a_{\pm} z, \text{ if } \alpha = 0,$$

where c_1 is a constant, and $a_{\pm} = \frac{-\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}$.

Theorem B If $\beta = 0$, then the general solution of (1.2) is given by

$$\omega(z) = \begin{cases} c_1 \exp(\pm \mathrm{i}\frac{\alpha z}{\sqrt{\gamma}}) - \frac{\gamma}{\alpha} & , \text{ if } \alpha \neq 0; \\ c_1 \pm \mathrm{i}\sqrt{\gamma} z & , \text{ if } \alpha = 0; \\ \frac{1}{c_1^2}[\alpha + \sqrt{\alpha^2 + \gamma c_1^2} \cosh(c_1 z + c_2)] & , \text{ where } c_1 \neq 0; \\ -\frac{\alpha}{2} z^2 + c_2 \alpha z - \frac{\gamma + c_2^2 \alpha^2}{2\alpha} & , \text{ if } \alpha \neq 0, \end{cases}$$

where c_1, c_2 are constants.

Theorem C If $\alpha = \gamma = 0$, then the general solution of (1.2) is given by

$$\omega(z) = \begin{cases} c_1 e^{c_2 z} + \frac{\beta}{c_2} & , \\ -\beta z + c_1 & , \\ 0 & , \end{cases}$$

where c_1, c_2 are constants.

However, their proofs are complicated. In this note, by using a new method, we give a unified and simplified proof for these known results. Specifically, our main results can be stated as follows:

Theorem 1.1 If $\gamma \neq 0$, consider the solutions of (1.2), we would have

$$\omega(z) = \begin{cases} c \exp(\frac{\alpha z}{a_{\pm}}) - \frac{\gamma}{\alpha} & , \quad \alpha \neq 0; \\ a_{\pm} z + c & , \quad \alpha = 0, \end{cases}$$

where c is a constant, and $a_{\pm} = \frac{-\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}$.

Theorem 1.2 If $\gamma = 0$, consider the solutions of (1.2), we would have

(1) If $\alpha\beta \neq 0$, then $\omega(z) = ce^{-\frac{\alpha}{\beta} z}$, here c is a constant;

(2) If $\alpha = 0$, then

$$\omega(z) = \begin{cases} c_1 e^{c_2 z} + \frac{\beta}{c_2} & , \\ -\beta z + c_1, & , \\ c_1 & , \end{cases}$$

where $c_1,\ c_2$ are constants.

(3) If $\beta = 0$, then

$$\omega(z) = \begin{cases} c_1 e^{\sqrt{A}z} + c_2 e^{-\sqrt{A}z} - \frac{\alpha}{A} & , \quad A \neq 0; \\ -\frac{\alpha}{2}z^2 + c_1 z + c_2 & , \quad A = 0, \end{cases}$$

where c_1, c_2 are constants such that $c_1^2 + 2\alpha c_2 = 0$ if $A = 0$ or $4c_1c_2A^2 = \alpha^2$ if $A \neq 0$.

2 Lemmas and Proofs of Theorems

The following lemma is crucial to the proof of our theorems.

Lemma 2.1 [3]. Let f be a meromorphic solution of an algebraic equation

$$P(z, f, f', \cdots, f^{(n)}) = 0, \tag{2.1}$$

where P is a polynomial in $f, f', \cdots, f^{(n)}$ with meromorphic coefficients small with respect to f. If a complex constant c does not satisfy equation (2.1), then

$$m(r, \frac{1}{f-c}) = S(r, f).$$

In order to prove the results, we also need the following lemma.

Lemma 2.2 [4] . Let h be a non-constant entire function, and $f = \mathrm{e}^h$, then

$$T(r, h) = o(T(r, f)), \quad T(r, h') = S(r, f).$$

Proof of Theorem 1.1.

Since $\gamma \neq 0$, then (1.2) and Lemma 2.1 imply

$$m(r, \frac{1}{\omega}) = S(r, \omega). \tag{2.2}$$

By the Nevanlinna's first fundamental theorem and (2.2), we get

$$N(r, \frac{1}{\omega}) = T(r, \omega) + S(r, \omega),$$

which, with (1.2), gives

$$N_{1)}(r, \frac{1}{\omega}) = T(r, \omega) + S(r, \omega), \tag{2.3}$$

where $N_{1)}(r, \frac{1}{\omega})$ denotes the counting function corresponding to simple zeros of ω.

Let $\omega(z_0) = 0$, then z_0 is a zero of $(\omega')^2 + \beta\omega' + \gamma$, and thus

$$(\omega'(z_0) - a_+)(\omega'(z_0) - a_-) = 0$$

with $a_+ = \frac{-\beta + \sqrt{\beta^2 - 4\gamma}}{2}$, $a_- = \frac{-\beta - \sqrt{\beta^2 - 4\gamma}}{2}$.

First, we assume that $\omega'(z_0) - a_+ = 0$, and set

$$h_1 = \frac{\omega' - a_+}{\omega}.$$

Next we will show that h_1 is a constant.

To prove this assertion, we first prove h_1 is a small function of ω. In fact, (2.2) and (2.3) give $m(r, h_1) = S(r, \omega)$.

By considering the order of any pole of ω in (1.2), we can check that there exists no possibility for such a pole, as shown in [9] and therefore all solutions are entire, and $N_{(2}(r, \frac{1}{\omega}) = S(r, \omega)$, where $N_{(2}(r, \frac{1}{\omega})$ denotes the counting function corresponding to multiple zeros of ω. Thus, $N(r, h_1) = S(r, \omega)$, and $T(r, h_1) = S(r, \omega)$.

Moreover, it follows by the definition of h_1 that

$$\omega' = h_1\omega + a_+, \quad \omega'' = (h_1' + h_1^2)\omega + h_1 a_+,$$

which, with (1.2), gives $h_1' \equiv 0$, and $h_1 = \frac{\alpha}{a_-}$. Thus, $\omega' - a_+ = \frac{\alpha}{a_-}\omega$, and we have $\omega(z) = c\exp(\frac{\alpha z}{a_-}) - \frac{\gamma}{\alpha},$ if $\alpha \neq 0$ or $\omega = a_+ z + c$, if $\alpha = 0$, where c is a constant.

If $\omega'(z_0) - a_- = 0$, we set

$$h_2 = \frac{\omega' - a_-}{\omega}.$$

In the same way, we get $\omega = c\exp(\frac{\alpha z}{a_+}) - \frac{\gamma}{\alpha}$, if $\alpha \neq 0$ or $\omega = a_- z + c$, if $\alpha = 0$, where c is a constant.

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2.

To prove Theorem 1.2, now we distinguish three cases to discuss.

Case 1. $\alpha\beta \neq 0$. Since $\gamma = 0$, it follows by (1.2) that

$$\omega\omega''' - \omega'\omega'' = \alpha\omega' + \beta\omega'',$$

this gives

$$(\omega'' + \alpha)(\omega')^2 = \omega\omega'\omega''' - \beta\omega'\omega''. \tag{2.4}$$

From (1.2) and (2.4), we get

$$\omega[(\omega'')^2 - \omega'\omega''' - \alpha^2] = \alpha\beta\omega'. \tag{2.5}$$

Note that all the solutions of (1.2) are entire functions, by (2.5), we see that $\omega \equiv 0$ or $\omega = e^h$, in which h is an entire function.

Substituting $\omega = e^h$ into (2.5), we have

$$\{[(h')^2 + h'']^2 - h'[3h'h'' + (h')^3 + h''']\}e^{2h} = \alpha^2 + \alpha\beta h'. \tag{2.6}$$

By the standard Valiron-Mohon'ko lemma (see, e.g., [10]), Lemma 2.2 and (2.6), we obtain $h' = -\frac{\alpha}{\beta}$ and so

$$\omega = ce^{-\frac{\alpha}{\beta}z},$$

where c is a constant.

Case 2. $\alpha = 0$. In this case, (1.2) gives $\omega\omega'' = \omega'(\omega' + \beta)$, thus $\omega = c_1$, or $\omega = -\beta z + c_1$, or

$$\frac{\omega''}{\omega' + \beta} = \frac{\omega'}{\omega},$$

and so

$$\omega = c_1 e^{c_2 z} + \frac{\beta}{c_2},$$

where c_1, c_2 are constants.

Case 3. $\beta = 0$. From (1.2), we conclude

$$[(\frac{\omega'}{\omega})^2]' = -2\alpha(\frac{1}{\omega})',$$

this leads to

$$(\frac{\omega'}{\omega})^2 = -\frac{2\alpha}{\omega} + A, \qquad (2.7)$$

where A is a constant.

Again, by (1.2) and (2.7) we find

$$\omega'' - A\omega + \alpha = 0,$$

which gives

$$\omega = \begin{cases} c_1 e^{\sqrt{A}z} + c_2 e^{-\sqrt{A}z} - \frac{\alpha}{A} & , \quad A \neq 0; \\ -\frac{\alpha}{2}z^2 + c_1 z + c_2 & , \quad A = 0, \end{cases}$$

where c_1, c_2 are constants such that $c_1^2 + 2\alpha c_2 = 0$ if $A = 0$ or $4c_1c_2A^2 = \alpha^2$ if $A \neq 0$.

This completes the proof of Theorem 1.2.

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Competing Interests

The authors declare that no competing interests exist.

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