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Study of exp(-Φ(ξ))-expansion Method for Solving Nonlinear Partial Differential Equations

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Abstract

In this work we study the Gardner equation or the combined KdV-mKdV equation. We use the exp(-Φ(ξ))-expansion method for a reliable treatment to establish exact traveling wave solutions then the solitary wave solutions for the aforementioned nonlinear partial differential equation (NPDEs). As a result, the traveling wave solutions are obtained in four arbitrary functions including hyperbolic function solutions, trigonometric function solutions, exponential function solutions, and rational function solutions.

Keywords: exp(-Φ(ξ))-expansion method, the Gardner equation, NPDEs, exact solutions.

MSC Numbers: 35C07, 35C08, 35P99.

1 Introduction

Mathematical modeling of dynamical processes in a great variety of natural phenomena leads in general to NPDEs. There is a particular class of solutions for these nonlinear equations that are of considerable interest. They are the traveling wave solutions such a wave is special solution of the

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governing equations, that may be localized or periodic, which does not change its shape and which propagates at constant speed. In the case of linear equations the profile is usually arbitrary. In contrast a nonlinear equation will normally determine a restricted class of profiles, as the result of a balance between nonlinearity and dissipation.

In recent years, the exact solutions of NPDEs have been investigated by many authors (see for example [1-23]) who are interested in nonlinear physical phenomena which exist in all fields including either the scientific works or engineering fields. Gardner equation is a nonlinear partial equation differential equation set up by mathematician Glifford Gardner in 1968 to generalized KdV equation. Gardner equation has application in hydrodynamics, plasma physics and quantum field theory etc.

The research of traveling wave solutions of some nonlinear evolution equations derived from such fields played an important role in the analysis of some phenomena, such as the exp(-Φ(ξ))-expansion method [1-3], the extended tanh-function method [4-5], the complex hyperbolic function method [6], the F-function expansion method [7], the exp-function expansion method [8], the Bernoulli's Sub-ODE method [9], the extended sinh-cosh and sin-cos methods [10], the modified simple equation method [11,12], the (G'/G)-expansion method [14-16], the enhance(G'/G)-expansion method [17] , Lie symmetry method [18-21] and so on.

The purpose of this article is to demonstrate the so called exp(-Φ(ξ))-expansion method for finding the exact traveling wave solutions of the Gardner equation or the combined KdV-mKdV equation which is very important in applied sciences.

The paper is arranged as follows: In section 2, the exp(-Φ(ξ))-expansion method is discussed; In section 3, we apply this method to the nonlinear Gardner equation; in section 4 graphical representations and in section 5 conclusions are given.

2 Methodology

Considering the nonlinear partial differential equation in the form

$$F(u,u_t,u_x,u_{tt},u_{xx},u_{ttt},u_{xxx}.......)=0. \tag{2.1}$$

where $u(\xi)=u(x,t)$ is an unknown function, F is a polynomial of $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method [1-3]:

Step 1. Combining the independent variables x and t into one variables $\xi = x \pm \omega t$, we suppose that

$$u(x,t)=u(\xi) \qquad \xi = x \pm \omega t \tag{2.2}$$

The traveling wave transformation equation (2.2) permits us to reduce eq. (2.1) to the following ordinary differential equation (ODE)

$$P(u, u', u'', \ldots\ldots\ldots) = 0. \tag{2.3}$$

where P is a polynomial in $u(\xi)$ and its derivatives, whereas $u'(\xi) = \frac{du}{d\xi}, u''(\xi) = \frac{d^2u}{d\xi^2}$, and so on.

Step 2. We suppose that Eq. (2.3) has the formal solution

$$u(\xi) = \sum_{i=0}^{n} A_i \left(\exp(-\Phi(\xi))\right)^i, \tag{2.4}$$

Where A_i $(0 \le i \le n)$ are constants to be determined, such that $A_n \ne 0$, and $\Phi = \Phi(\xi)$ satisfies the following ODE

$$\Phi'(\xi) = \exp(-\Phi(\xi)) + \mu \exp(\Phi(\xi)) + \lambda. \tag{2.5}$$

Eq. (2.5) gives the following solutions:
When $\lambda^2 - 4\mu > 0, \mu \ne 0,$

$$\Phi(\xi) = \ln\left(\frac{-\sqrt{(\lambda^2 - 4\mu)} \tanh\left(\frac{\sqrt{(\lambda^2 - 4\mu)}}{2}(\xi + k)\right) - \lambda}{2\mu}\right), \tag{2.6}$$

$$\Phi(\xi) = \ln\left(\frac{-\sqrt{(\lambda^2 - 4\mu)} \coth\left(\frac{\sqrt{(\lambda^2 - 4\mu)}}{2}(\xi + k)\right) - \lambda}{2\mu}\right), \tag{2.7}$$

When $\lambda^2 - 4\mu < 0, \mu \ne 0,$

$$\Phi(\xi) = \ln\left(\frac{\sqrt{(4\mu - \lambda^2)} \tan\left(\frac{\sqrt{(4\mu - \lambda^2)}}{2}(\xi + k)\right) - \lambda}{2\mu}\right), \tag{2.8}$$

$$\Phi(\xi) = \ln\left(\frac{-\sqrt{(4\mu-\lambda^2)}\cot\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\xi+k)\right)-\lambda}{2\mu}\right), \tag{2.9}$$

When $\lambda^2 - 4\mu > 0, \mu = 0, \lambda \neq 0$,

$$\Phi(\xi) = -\ln\left(\frac{\lambda}{\exp(\lambda(\xi+k))-1}\right), \tag{2.10}$$

When $\lambda^2 - 4\mu = 0, \mu \neq 0, \lambda \neq 0$,

$$\Phi(\xi) = \ln\left(-\frac{2(\lambda(\xi+k)+2)}{\lambda^2(\xi+k))}\right), \tag{2.11}$$

When $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0$,

$$\Phi(\xi) = \ln(\xi + k) \tag{2.12}$$

where k is an arbitrary constant and $A_n, \cdots\cdots, \omega, \lambda, \mu$ are constants to be determined later, $A_n \neq 0$, the positive integer n can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (2.3).

In [1-3], the Eq. (2.5) has five solutions, but in this paper we have used seven solutions. Two solutions are newly explored by us which are absent in the previous published papers.

Step 3. We substitute Eq. (2.4) into (2.3) and then we account the function exp(-Φ(ξ)). As a result of this substitution, we get a polynomial of exp(-Φ(ξ)). We equate all the coefficients of same power of exp (-Φ(ξ)) to zero. This procedure yields a system of algebraic equations whichever can be solved to find $A_n, \cdots\cdots, \omega, \lambda, \mu$. Substituting the values of $A_n, \cdots\cdots, \omega, \lambda, \mu$ into Eq. (2.4) along with general solutions of Eq. (2.5) completes the determination of the solution of Eq. (2.1).

3 Application

In this section, we will apply the exp(-Φ(ξ))-expansion method to find the exact solutions and then the solitary wave solutions of Gardner equation or the combined KdV-mKdV equation,

$$u_t = 6uu_x + 6\varepsilon^2 u^2 u_x + u_{xxx}, \tag{3.1}$$

where $u(x,t)$ is the amplitude of the relevant wave mode and ε is a nonzero constant. The terms u_t is the time evolution of the wave propagation, uu_x and u^2u_x are the nonlinearity, that accounts for steepening of the wave and u_{xxx} represents linear dispersion, that describes the spreading of the wave. The Gardner equation was first derived rigorously within the asymptotic theory for long internal waves in a two-layer fluid with a density jump at the interface [22,23]. The Gardner equation is widely used in various branches of physics, such as plasma physics, fluid physics, quantum field theory. The equation plays a prominent role in ocean waves. The Gardner equation describes internal waves and admits interesting solutions.

The traveling wave transformation

$$u = u(x,t), \xi = x + \omega t, u = u(\xi), u(x,t) = u(\xi), \tag{3.2}$$

permits us to transform Eq. (3.1) into the following ordinary differential equation:

$$\omega u' + 6uu' + 6\varepsilon^2 u^2 u' + u''' = 0. \tag{3.3}$$

Integrating with respect to ξ, we obtain the following ODE

$$C + \omega u + 3u^2 + 2\varepsilon^2 u^3 + u'' = 0. \tag{3.4}$$

Balancing the order of u^3 and u'' in Eq. (3.4), we get $3n = n + 2$, which gives $n = 1$.
Hence for $n = 1$ Eq. (2.4) becomes

$$u(\xi) = A_0 + A_1(\exp(-\Phi(\xi))). \tag{3.5}$$

where A_0 and A_1 are constants to be determined later such that $A_1 \neq 0$ while λ and μ are arbitrary constants.

Substituting u, u^2, u^3 and u'' into Eq. (3.4) and then equating the coefficients of exp(-Φ(ξ)) to zero, yields a set of simultaneous algebraic equations as follows:

$$2\varepsilon^2 A_0^3 + 3A_0^2 + A_1\mu\lambda + C + \omega A_0 = 0, \tag{3.6}$$

$$2A_1\mu + \omega A_1 + 6\varepsilon^2 A_0^2 A_1 + 6A_0A_1 + A_1\lambda^2 = 0, \tag{3.7}$$

$$6\varepsilon^2 A_0 A_1^2 + 3A_1\lambda + 3A_1^2 = 0, \tag{3.8}$$

$$2\varepsilon^2 A_1^3 + 2A_1 = 0, \tag{3.9}$$

Solving the algebraic equations above yields

$$C=-\frac{1}{4}\frac{4\mu\varepsilon^2-1-\lambda^2\varepsilon^2}{\varepsilon^4},\quad \omega=-\frac{1}{2}\frac{1-\lambda^2\varepsilon^2+4\mu\varepsilon^2-3}{\varepsilon^2},\quad A_0=\frac{1}{2}\left(\frac{\pm I\varepsilon\lambda-1}{\varepsilon^2}\right),\quad A_1=\pm\frac{I}{\varepsilon}$$

where λ and μ are arbitrary constants and ε is a nonzero constant.
Also $I=\sqrt{-1}$.

Now putting the values of C,ω,A_0 and A_1 into Eq. (3.5) yields

$$u(\xi)=\frac{\pm I}{\varepsilon}\left(\frac{\lambda}{2}+\exp(-\Phi(\xi))\right)-\frac{1}{2\varepsilon^2}\ , \tag{3.10}$$

where $\xi=x+\dfrac{1}{2}\dfrac{\left(-\lambda^2\varepsilon^2+4\mu\varepsilon^2-3\right)t}{\varepsilon^2}$,

Now substituting Eq. (2.6)-Eq. (2.12) into Eq. (3.10) respectively, yields a set of traveling wave solutions of the Gardner equation.

When $\lambda^2-4\mu>0,\mu\neq 0,$

$$u_{1,2}(\xi)=\frac{1}{2}\left(\frac{\pm I\varepsilon\lambda-1}{\varepsilon^2}\right)\pm\frac{2Iu}{\left(\varepsilon\left(-\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}(\xi+k)\right)-\lambda\right)\right)},$$

$$u_{3,4}(\xi)=\frac{1}{2}\left(\frac{\pm I\varepsilon\lambda-1}{\varepsilon^2}\right)\pm\frac{2Iu}{\left(\varepsilon\left(-\sqrt{\lambda^2-4\mu}\coth\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}(\xi+k)\right)-\lambda\right)\right)},$$

where $\xi=x+\dfrac{1}{2}\dfrac{\left(-\lambda^2\varepsilon^2+4\mu\varepsilon^2-3\right)t}{\varepsilon^2}$ and k is an arbitrary constant.

When $\lambda^2-4\mu<0,\mu\neq 0,$

$$u_{5,6}(\xi)=\frac{1}{2}\left(\frac{\pm I\varepsilon\lambda-1}{\varepsilon^2}\right)\pm\frac{2Iu}{\left(\varepsilon\left(\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}(\xi+k)\right)-\lambda\right)\right)},$$

$$u_{7,8}(\xi)=\frac{1}{2}\left(\frac{\pm I\varepsilon\lambda-1}{\varepsilon^2}\right)\pm\frac{2I\mu}{\left(\varepsilon\left(\sqrt{4\mu-\lambda^2}\cot\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}(\xi+k)\right)-\lambda\right)\right)},$$

where $\xi=x+\frac{1}{2}\frac{\left(-\lambda^2\varepsilon^2+4\mu\varepsilon^2-3\right)t}{\varepsilon^2}$ and k is an arbitrary constant.

When $\lambda^2-4\mu>0, \mu=0, \lambda\neq 0$,

$$u_{9,10}(\xi)=\frac{1}{2}\left(\frac{\pm I\varepsilon\lambda-1}{\varepsilon^2}\right)\pm\frac{I\lambda}{\varepsilon(\exp(\lambda(\xi+k))-1)},$$

where $\xi=x+\frac{1}{2}\frac{\left(-\lambda^2\varepsilon^2-3\right)t}{\varepsilon^2}$ and k is an arbitrary constant.

When $\lambda^2-4\mu=0, \mu\neq 0, \lambda\neq 0$,

$$u_{11,12}(\xi)=\frac{1}{2}\left(\frac{\pm I\varepsilon\lambda-1}{\varepsilon^2}\right)\pm\frac{\frac{1}{2}I\lambda^2\left(x-\frac{3}{2}\frac{t}{\varepsilon^2}+k\right)}{\varepsilon\left(\lambda\left(x-\frac{3}{2}\frac{t}{\varepsilon^2}+k\right)+2\right)},$$

where k is an arbitrary constant.

When $\lambda^2-4\mu=0, \mu=0, \lambda=0$,

$$u_{13,14}(\xi)=-\frac{1}{2\varepsilon^2}\pm\frac{I}{\varepsilon\left(x-\frac{3t}{2\varepsilon^2}+k\right)},$$

where k is an arbitrary constant.

4 Graphical Representation of some obtained Solutions

Using mathematical software Maple 13, 2D and 3D plots of some obtained solutions have been shown in Figs. 1- 4 to visualize the underlying mechanism of the original equations.

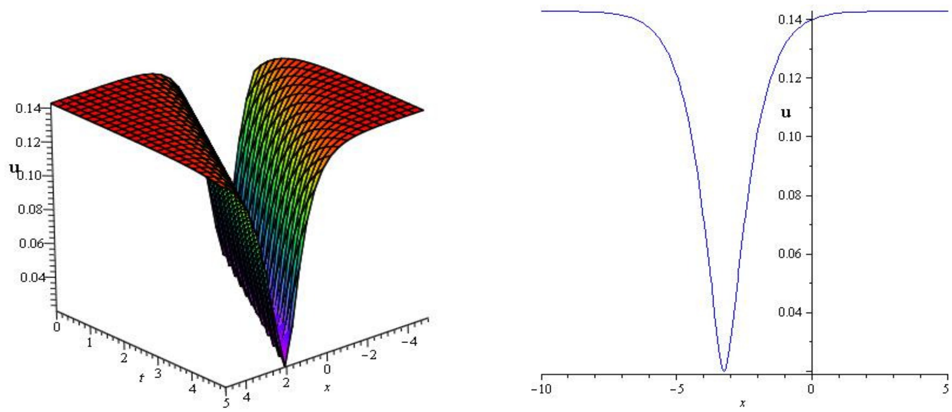

Fig. 1. Dark (black) soliton profile of Gardner equation for $\lambda = 2, \mu = 0.50,$ $k = 0, \varepsilon = 5$. (Only shows the shape of $u_3(\xi)$), The left figure shows the 3D plot and the right figure shows the 2D plot for *t*=0

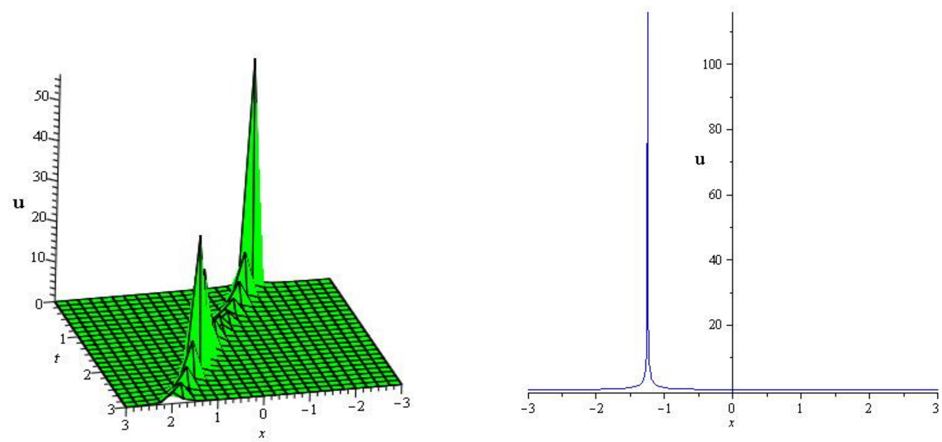

Fig. 2. Soliton profile of Gardner equation for $\lambda = 2, \mu = 0.50,$ $k = 0, \varepsilon = 5$ (Only shows the shape of $u_3(\xi)$). The left figure shows the 3D plot and the right figure shows the 2D plot for *t*=0

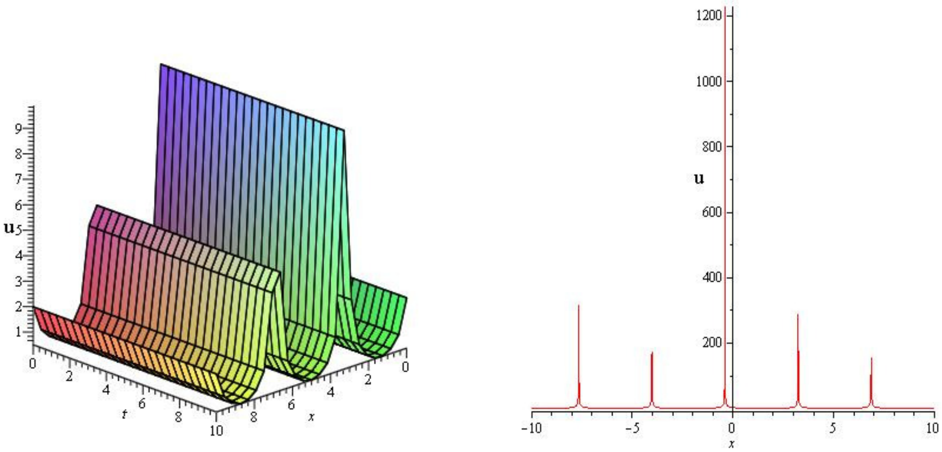

Fig. 3. Soliton profile of Gardner equation for $\lambda = 1, \mu = 1, k = 1, \varepsilon = 1$ (only shows the shape of $u_5(\xi)$), The left figure shows the 3D plot and the right figure shows the 2D plot for *t*=0

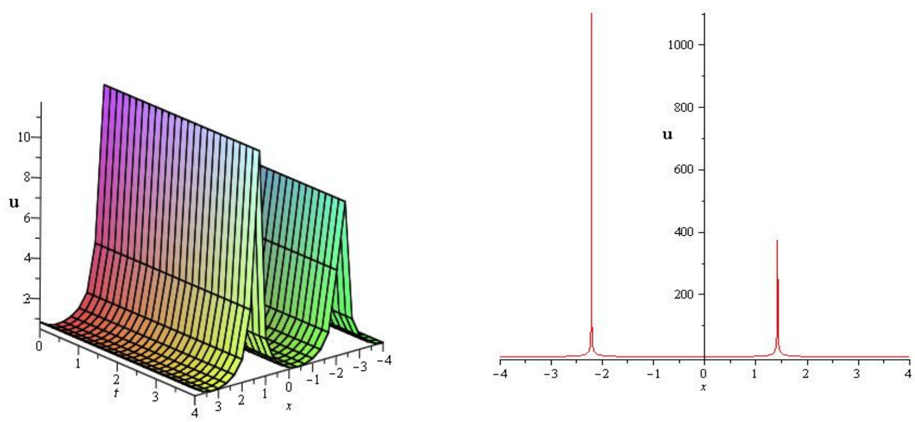

Fig. 4. Periodic wave profile of Gardner equation for $\lambda = 1, \mu = 1, k = 2, \varepsilon = 2$. (Only shows the shape of $u_7(\xi)$), The left figure shows the 3D plot and the right figure shows the 2D plot for *t*=0

5. Conclusion

From this study, we have seen that the traveling wave solution in terms of hyperbolic, trigonometric and rational functions for the Gardner equation are successfully obtained by using the exp(-Φ(ξ))-expansion method. Finally, the solutions of the nonlinear evolution equation obtained in this paper have many potential applications in mathematical physics and engineering. The presented approach is applicable for similar nonlinear equations.

Competing Interests

Authors have declared that no competing interests exist.

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