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On New Identities for Basic Polynomials Sequences in Finite Operator Calculus

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Authors' contributions

This work was carried out in collaboration between both authors. Author AM designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript and managed literature searches. Author CE managed the analyses of the study and literature searches. Both authors read and approved the final manuscript.

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Abstract

The main objective of this paper is to investigate the characterization of the delta operator for some basic polynomials. A theorem from G.C.Rota which gives the necessary and sufficient conditions for the sequence of basic polynomials corresponding to some delta operator Q is reconstructed in terms of three new identities and this theorem is verified by some comprehensive examples.

Keywords: Delta operator; basic polynomial sequences; binomial type sequences.

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1 Introduction

G. C. Rota [1] serves as an introduction and a guide to the growing literature of combinatorics. It contains a detailed study of the delta operator and their basic polynomials. Octavian Agratini [2] contains a similar study which applies binomial sequences in the construction of linear approximation processes. Miloud Mihoubi [3] derived some relations between the sequences of Binomial type and Bell polynomials. Several classical polynomials and their properties are discussed in Rainville, E. D. [4]. Maheswaran, A [5] is a study of discrete special polynomials in q-monodiffric sense through finite operator calculus.

The aim of the present paper is to propose some results tied to the basic polynomials corresponding to the delta operator Q. This rest of the paper is organized in five sections. In the second section, we give some known definitions and theorems from G.C.Rota [1]. In the third section, we discuss about the sequential expression for the delta operator Q. This sequential representation of the delta operator plays a vital role in deriving new identities for the sequence of basic polynomials. In the fourth section, the characterization of the delta operator Q is investigated for some basic polynomials. A theorem connecting the sequence of basic polynomials for some delta operator with the sequence of polynomials of binomial type is developed in G. C. Rota [1]. In the fifth section, this theorem is reconstructed independently in terms of three new identities. In the sixth section, the reconstructed theorem is verified through some comprehensive examples.

2 Preliminaries

The operational calculus had been known as early as the beginning of the nineteenth century, but its improvement was due to the later work of Heaviside, who applied it widely to problems in electricity. In 1927, J. F. Steffensen introduced the *theta operators*. In 1956, F.B. Hildbrand called them *delta operators* and this term was taken over and intensively used by G. C.Rota. Rota's operator approach to the finite operator calculus is a systematic study of delta operators on the algebra of polynomials. In this section, we recall terminology, notation, some basic definitions and results of the finite operator calculus, as it has been introduced by Rota [1].

Let F be a Field of characteristic zero, preferably the real number field. Let $p(x)$ be a polynomial in one variable defined over F. The set of such polynomials is denoted by P. A sequence of polynomials is $\{p_n(x)/n \in \mathbb{Z}^+ \cup \{0\}\}$, where $p_n(x)$ is exactly of degree n.

Definition 1

i An operator E^a is said to be a shift operator if $E^a p(x) = p(x+a)$, for all polynomials $p(x)$ in one variable defined over the field F and $a \in F$.

ii A linear operator T which commutes with all shift operators is called a shift invariant. In symbols, $TE^a = E^a T, \ \forall a \in F$.

iii A delta operator usually denoted by the letter Q, is a shift-invariant operator for which Qx is a non zero constant.

Thus every delta operator Q is a shift invariant. But a shift invariant operator need not be a delta operator.

The forward difference operator

$$(\Delta f)(x) = f(x+1) - f(x)$$

is a delta operator.

Any operator of the form

$$\sum_{k=1}^{\infty} c_k D^k, \ \textit{where } D^n(f) = f^n \textit{ is the } n^{th} \textit{ derivative with } c \neq 0$$

is a delta operator.

Definition 2.

Let Q be a delta operator, A polynomial sequence $p_n(x)$ is called the sequence of basic polynomials for Q if :

1. $p_0(x) = 1$
2. $p_n(0) = 0, \quad \textit{whenever } n > 0$
3. $Qp_n(x) = np_{n-1}(x)$

The basic polynomials are a large class of polynomial sequences that include the Monomials $\{x^n; n = 0, 1, 2, \cdots n\}$, the sequences of Lower factorials $[x]_n$, Upper factorials $[x]^n$, the Abel polynomials and many others.

Definition 3.

A polynomial sequence $p_n(x)$ is said to be a binomial type if it satisfies the infinite sequence of following identities

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)\ p_{n-k}(y), \quad n = 0, 1, 2, \ldots$$

The simplest sequence of Binomial type is $\{x^n\}$. Several properties of the polynomial sequence $\{x^n\}$ can be generalized to an arbitary sequence of basic polynomials. The most important sequence of binomial type is the sequence of Abel polynomials, namely, the sequence $p_n(x) = x(x + na)^{n-1}$ for $a \in Q$. This polynomials play a leading role in the theory of sequences of binomial type. The main result is that any sequence of binomial type can be represented as Abel polynomials.

The proofs of following results are skipped. But they are easily read from the reference G.C.Rota [1].

Theorem 1.

i) *If Q is a delta operator, then $Qa = 0$ for every constant $'a'$.*

ii) *If $p(x)$ is a polynomial of degree n, then $Qp(x)$ is a polynomial of degree $n - 1$.*

The delta operators possesses many of the properties of the usual derivative D. The above theorems are good examples.

Theorem 2.

Every delta operator has a unique sequence of basic polynomials.

Theorem 3.

(a) If $p_n(x)$ is a basic sequence for some delta operator Q, then it is a sequence of polynomials of Binomial type.

(b) If $p_n(x)$ is a sequence of polynomials of Binomial type, then it is a basic sequence for some delta operator.

Thus we have $p_n(x)$ is a basic polynomials sequence for some delta operator Q if and only if it is a sequence of polynomials of Binomial type.

According to G.C.Rota[1], Delta operator Q possesses many of the properties of usual derivative operator D. Generally usual derivative D is a delta operator. But the operator defined by $Q(x^n) = x^{n-1}$, $n \in \mathbb{Z}^+ \cup \{0\}$, will not be a delta operator , since it is not shift invariant.

Taking $Q(x^n) = a_n\ x^{n-1}$ where a_n is a real constant , for $n \in \mathbb{Z}^+$ and assuming Q to be a delta operator, $E^aQ(x^n) = QE^a(x^n)$ implies $a_n\binom{n-1}{r} = \binom{n}{r}\ a_{n-r}$ and hence $a_n = na_1$. In this case, the delta operator becomes a constant multiple of the usual derivative operator D. This leads to the theorem :

Theorem 4.

"If Q is a delta operator and $Q(x^n) = a_n\ x^{n-1}$, where a_n is a real constant , $n \in \mathbb{Z}^+$, then $Q = kD$ where k is a real constant and D is the usual derivative."

3 Sequential Representation of Delta Operator Q

Using the expressions for $Q(x^2), Q(x^3) \cdots Q(x^n)$, we attempt to formulate the delta operator in terms of a sequence of real numbers. By Theorem 1 and definition of the basic polynomials, we obtain the following Theorem.

Theorem 5. For the monomials $\{x^n :\ n \in \mathbb{Z}^+ \cup \{0\}\}$, and for each α_r an arbitary real value,

$$Q(x^n) = \sum_{r=1}^{n} \binom{n}{r}\ \alpha_r\ x^{n-r}. \tag{3.1}$$

Proof .

Taking $Q(x) = \alpha_1 \neq 0$ and construct $Q(x^2) = c_0\ x + c_1$. Since Q is shift invariant, we have $E^aQ(x^2) = QE^a(x^2)$. Solving we get $c_0 = 2\alpha_1$ and c_1 is a new independent constant which may be taken as α_2. Hence $Q(x^2) = 2\alpha_1 x + \alpha_2$. Thus the theorem is true for $n = 1$ and 2.

Let us assume that the result is true for all $n = k$.

Therefore ,

$$Q(x^k) = \sum_{r=1}^{k} \binom{k}{r}\ \alpha_r\ x^{k-r} = \binom{k}{1}\ \alpha_1\ x^{k-1} + \binom{k}{2}\ \alpha_2\ x^{k-2} + \cdots + \binom{k}{r}\ \alpha_r\ x^{k-r} + \cdots + \alpha_k \tag{3.2}$$

Since $\{x^n\}$ is a basic polynomial sequence, it satisfies $Qp_n(x) = np_{n-1}(x)$ and hence we have,

$$Q(x^k) = k\ x^{k-1} \tag{3.3}$$

From (3.3), we see that the delta operator Q is a usual derivative D.

From (3.2) and (3.3) ,

$$\binom{k}{1}\ \alpha_1\ x^{k-1} + \binom{k}{2}\ \alpha_2\ x^{k-2} + \cdots + \binom{k}{r}\ \alpha_r\ x^{k-r} + \cdots + \alpha_k = k\ x^{k-1} \tag{3.4}$$

By comparing the corresponding terms, we have $\alpha_1 = 1$ and $\alpha_j = 0, \ j = 2, 3, \cdots k$

Therefore, the result is true for $n = k$ means that

$$\alpha_1 = 1 \ and \ \alpha_j = 0 \ (j = 2, 3, \cdots k) \tag{3.5}$$

Now we have to show that this result is true for $n = k + 1$

$$Q(x^{k+1}) = Q(x^k \ x) = Q(x^k) \ x + Q(x) \ x^k \ = (k+1) \ x^k$$

Thus we have

$$Q(x^{k+1}) = (k+1) \ x^k \tag{3.6}$$

On other hand, using the property that $Qp_n(x) = n \ p_{n-1}(x)$, we have

$$Q(x^{k+1}) = (k+1) \ p_k(x) = (k+1) \ x^k \tag{3.7}$$

From the Equations (3.6) and (3.7), we conclude that the result is true for all $n = k + 1$

Thus we proved the Theorem 5.

Here, $Q(x^n)$ has n independent parameters, $\alpha_i, (i = 1, 2, 3 \ldots n)$. These parameters are unique. Allowing n being large, we get an infinite sequence of real numbers. Hence to represent a delta operator, we need to consider only the corresponding infinite sequence of real numbers . We conclude that any delta operator may be fixed uniquely by Equation (3.1). To study the delta operator, we need analyse only this sequential representation in Equation (3.1).

Below we list values of $Q(x^n)$ for $n \geq 1$

$$\begin{array}{c}
1\alpha_1 \\
2 \ \alpha_1 \ x + 1\alpha_2 \\
3 \ \alpha_1 \ x^2 + 3 \ \alpha_2 \ x + 1\alpha_3 \\
4 \ \alpha_1 \ x^3 + 6 \ \alpha_2 \ x^2 + 4 \ \alpha_3 \ x + 1\alpha_4 \\
5 \ \alpha_1 \ x^4 + 10 \ \alpha_2 \ x^3 + 10 \ \alpha_3 \ x^2 + 5 \ \alpha_4 \ x + 1\alpha_5 \\
6 \ \alpha_1 \ x^5 + 15 \ \alpha_2 \ x^4 + 20 \ \alpha_3 \ x^3 + 15 \ \alpha_4 \ x^2 + 6 \ \alpha_5 \ x + 1\alpha_6 \\
7 \ \alpha_1 \ x^6 + 21 \ \alpha_2 \ x^5 + 35 \ \alpha_3 \ x^4 + 35 \ \alpha_4 \ x^3 + 21 \ \alpha_5 \ x^2 + 7 \ \alpha_6 \ x^5 + 1\alpha_7
\end{array}$$

The coefficient of $Q(x^n)$ are arranged by a triangular array, say *delta triangle* is given below

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 2 & & 1 & & & & \\
 & & & 3 & & 3 & & 1 & & & \\
 & & 4 & & 6 & & 4 & & 1 & & \\
 & 5 & & 10 & & 10 & & 5 & & 1 & \\
6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\ldots & & & & & & & & & &
\end{array}$$

Similar to Pascal triangle, it is also a triangular arrangements of rows. The tip of the triangle is number 1 which makes up the first row. In Pascal triangle, each row, except first, begins and ends with a "1". But in delta triangle, the consecutive rows begins with numbers 1,2,3,... respectively but ending with 1s. From the second row, the "Pascal Triangle sum" result holds good.

Equation (3.1) in Theorem 5 is important in deriving many results in the further sections. The characterization of the delta operator is determined by the values of $\alpha_i's$ $(i = 1, 2, 3 \cdots n)$. In the next section, we study more about the delta operator in particular, the characterization of a delta operator which corresponds to a given sequence of basic polynomials.

4 Finding Delta Operator for Given Basic Polynomial Sequence

For each delta operator assigned, there exists a unique sequence of basic polynomials. This basic sequence is obtained by using Theorem (5). This is done in [6]. The converse is interesting. A sequence of basic polynomials is given. Correspondingly can we get a unique delta operator ?. This is answered in this section.

Taking a set of polynomials $p_n(x)$ satisfying $p_0(x) = 1$ and $p_n(0) = 0,$ *whenever* $n > 0.$
This polynomials set may be considered to be a basic set. For this purpose, this set has to satisfy $Qp_n(x) = np_{n-1}(x)$; here Q is unknown. This unknown is detected from solving $Qp_n(x) = np_{n-1}(x)$ for all $n \geqslant 1$. Our aim is to check whether this polynomials set is a basic set and if so, find the corresponding the delta operator.

By the sequential representation of Q in Equation(3.1) in Theorem (5) and definition of sequence of basic polynomials, we obtain the following propositions.

Proposition 1. For a basic polynomials sequence $p_n(x) = \{x^n/n \in \mathbb{Z}^+ \cup \{0\}\}$, the characterization of the delta operator being $\alpha_1 = 1,$ and $\alpha_r = 0$ for all $r \geq 2$.

Proof. Let $p_n(x) = x^n$, $n \in \mathbb{Z}^+ \cup \{0\}$.
It satisfies $p_0(x) = 1$ and $p_n(0) = 0,$ whenever $n > 0$. Hence it is a basic set .
Using $Qp_n = np_{n-1}$ for $n = 1$, $Qp_1 = 1p_0 \ \Rightarrow\ Q(x) = 1$
But $Q(x) = \alpha_1$ from (3.1) $\ \Rightarrow\ \alpha_1 = 1$
Using $Qp_n = np_{n-1}$ for $n = 2$, $Qp_2 = 2p_1 \ \Rightarrow\ Q(x^2) = 2x$
But $Q(x^2) = 2\alpha_1 x + \alpha_2$ from (3.1) $\ \Rightarrow\ \alpha_1 = 1$ and $\alpha_2 = 0$
Using $Qp_n = np_{n-1}$ for $n = 3$, $Qp_3 = 3p_2 \ \Rightarrow\ Q(x^3) = 3x^2$
But $Q(x^3) = 3\alpha_1 x^2 + 3\alpha_2 x + \alpha_3$ from (3.1) $\ \Rightarrow\ \alpha_1 = 1,\ \alpha_2 = 0$ and $\alpha_3 = 0$
By similar procedure, we get $\alpha_4 = 0,\ \alpha_5 = 0 \cdots \alpha_n = 0$
Hence the characterization of the delta operator for $p_n(x) = \{x^n/n \in \mathbb{Z}^+ \cup \{0\}\}$ being $\alpha_1 = 1,$ and $\alpha_r = 0$ for all $r \geq 2$.

Remark 1. Here, $Q(x^n) = nx^{n-1}$ and the delta operator Q is usual derivative D.

By above Proposition (1), we obtain the following corollary.

Corollary 1. For any real constant k and a sequence of basic polynomials $p_n(x) = \{\frac{x^n}{k^n}/n \in \mathbb{Z}^+ \cup \{0\}\}$, the characterization of the delta operator being $\alpha_1 = k,$ and $\alpha_r = 0$ for all $r \geq 2$.

Remark 2. Here, $Q(x^n) = k\ n\ x^{n-1}$ and the delta operator Q is a constant multiple of the usual derivative D.

G. C. Rota [1] define the Difference polynomials (also called as Falling factorial polynomials) as follows :

$$p_n(x) = [x]_n = x(x-1)(x-2)\cdots(x-n+1)$$

Again by our main result in Equation (3.1), we obtain the following Proposition.

Proposition 2. For the Difference polynomials $p_n(x) = [x]_n = x(x-1)(x-2)\cdots(x-n+1)$, the characterization of the delta operator being $\alpha_n = 1$ for all $n \geq 1$.

proof . Let $p_n(x) = [x]_n$

It satisfies $p_0(x) = 1$ *and* $p_n(0) = 0$, *whenever* $n > 0$. Hence it is a basic set.

The first few polynomials are :

$p_1(x) = x$
$p_2(x) = x(x-1)$
$p_3(x) = x(x-1)(x-2)$
$p_4(x) = x(x-1)(x-2)(x-3)$ and so on.
For $n = 1$, $Qp_n = np_{n-1}$ becomes $Qp_1 = 1p_0$
From(3.1), $Qp_1 = \alpha_1$ and $1p_0 = 1 \Rightarrow \alpha_1 = 1$
For $n = 2$, $Qp_n = np_{n-1}$ becomes $Qp_2 = 2p_1$
By (3.1), $Qp_2 = 2\alpha_1 x + \alpha_2 - \alpha_1$ and $Qp_2 = 2p_1 \Rightarrow \alpha_1 = 1$ and $\alpha_2 = 1$
For $n = 3$, $Qp_n = np_{n-1}$ becomes $Qp_3 = 3p_2$
By (3.1), $Qp_3 = 3\alpha_1 x^2 + 3\alpha_2 x - 6\alpha_1 x + \alpha_3 - 3\alpha_2 + 2\alpha_1$ and $3p_2 = 3x^2 - 3x$
Equating the corresponding terms, we get $\alpha_1 = 1$, $\alpha_2 = 1$ and $\alpha_3 = 1$
Similarly proceeding as above we get, $\alpha_n = 1$, for all $n \geq 1$
Hence the sequential characterization of the delta operator being $\alpha_n = 1$, for all $n \geq 1$

Remark 3. Here, $Q(x^n) = \sum_{r=1}^{n} \binom{n}{r} x^{n-r}$. The same polynomials $p_n(x) = [x]_n$ are discussed in [3]. A Proposition which gives some relations between Bell polynomials and the sequences of binomial type is derived by Miloud Mihoubi [3] and this Proposition is verified through the polynomials $p_n(x) = [x]_n$.

The Abel polynomials are defined by

$$p_n(x) = x\ (x - na)^{n-1}$$

The sequence of Abel polynomials is most important sequence of binomial type and associated to the operator De^{aD}, refer [7]. The operator De^{aD} satisfies Definition (2) for Abel polynomials and e^{aD} is equivalent with E^a. But the characterization of the delta operator Q for the Abel polynomials is interesting and it is obtain in the following Proposition.
For $a = 1$, the Abel polynomials are

$$p_n(x) = x\ (x - n)^{n-1}$$

Proposition 3. For the Abel polynomials of first kind $p_n(x) = x\ (x-n)^{n-1}$, the characterization of delta operator being $\alpha_n = n$ for all $n \geq 1$.

proof . Let $p_n(x) = x\ (x-n)^{n-1}$.

It satisfies $p_0(x) = 1$ *and* $p_n(0) = 0$, *whenever* $n > 0$. Hence it is a basic set.

The first few polynomials are :

$$p_1(x) = x, \quad p_2(x) = x(x-2), \quad p_3(x) = x(x-3)^2, \quad p_4(x) = x(x-4)^3, \ \textit{and so on.}$$

For $n = 1$, $Qp_n = np_{n-1} \Rightarrow Qp_1 = 1p_0$
From(3.1), $Qp_1 = \alpha_1$ and $1p_0 = 1$
Comparing the corresponding terms, we get $\alpha_1 = 1$
For $n = 2$, $Qp_n = np_{n-1} \Rightarrow Qp_2 = 2p_1$
From (3.1), $Qp_2 = 2\alpha_1 x + \alpha_2 - 2\alpha_1$ and $2p_1 = 2x$
Comparing the corresponding terms, we get $\alpha_1 = 1$ and $\alpha_2 = 2$
For $n = 3$, $Qp_n = np_{n-1} \Rightarrow Qp_3 = 3p_2$
$Qp_3 = (3\alpha_1)x^2 + (3\alpha_2 - 12\alpha_1)x + 9\alpha_1 - 6\alpha_2 + \alpha_3$ and $3p_2 = 3x^2 - 6x$

Comparing the corresponding terms, we get $\alpha_1 = 1, \quad \alpha_2 = 2$ and $\alpha_3 = 3$
By Similar procedure as above, we get $\alpha_4 = 4$, $\alpha_5 = 5$, $\cdots \ \alpha_n = n$,
Hence the sequential characterization of the delta operator being $\alpha_n = n$, for all $n \geq 1$

Remark 4. For the above Abel polynomials, $Q(x^n) = \sum_{r=1}^{n} n\binom{n}{r}\ x^{n-r}$.

By above Proposition 3, we obtain the following Corollary.

Corollary 2. For the Abel polynomials of second kind $p_n(x) = x\ (x+n)^{n-1}$, the characterization of delta operator being $\alpha_n = n$ if n is add and $\alpha_n = -n$ if n is even , where $n \geq 1$.

From the above discussion, we get a way opened to study the basic polynomials by a new approach of finding definite delta operator numerically. All the above results are shown vividly in the following Table.

Table 1. Delta operators for different basic polynomials

Polynomials	Characterization of Delta Operator
Monomial $\{x^n\}$	$\alpha_1 = 1$ and $\alpha_r = 0$, for all $r \geq 2$.
$p_n(x) = \{\frac{x^n}{k^n}\}$	$\alpha_1 = k$ and $\alpha_r = 0$, for all $r \geq 2$.
Falling Factorial	$\alpha_1 = 1$, for all $r \geq 1$.
Abel-First kind	$\alpha_r = r$ for all $r \geq 1$.
Abel-Second kind	$\alpha_r = r$ if r is add and $\alpha_r = -r$ if r is even , where $r \geq 1$.

5 New Identities for Basic Polynomials Sequences

A new form of Newton binomial is discussed in [8]. By this method, The Equation (3.1) can be written as :

$$Q(x^n) = \sum_{r=1}^{n} \frac{1}{r!}\ \alpha_r\ D^r\ x^n. \tag{5.1}$$

Therefore,

$$Q \equiv \sum_{r=1}^{n} \frac{1}{r!}\ \alpha_r\ D^r. \tag{5.2}$$

Putting $p_n(x) = p_n$ and $D^r p_n(x) = p_n^{(r)}$, we have,

$$Q(p_n) = \sum_{r=1}^{n} \frac{1}{r!}\ \alpha_r\ p_n^{(r)}. \tag{5.3}$$

By (3) in definition (2), the above Equation (5.3) becomes,

$$Q(p_n) = \sum_{r=1}^{n} \frac{1}{r!}\ \alpha_r\ p_n^{(r)} = np_{n-1}. \tag{5.4}$$

Our main result in Equation (3.1) is used to derive some new identities for basic polynomial sequence.

Let $p_1(x) = c_{1,1}\ x$.
Rota[1] proved the Theorem 3, by omitting the constant term.
For $n = 1$, $Qp_n = np_{n-1}$ becomes $Qp_1(x) = 1p_0(x) \Rightarrow c_{1,1}Q(x) = 1$
Since $Q(x) = \alpha_1$ from eqn(3.1), we get

$$c_{1,1}\ \alpha_1 = 1. \tag{5.5}$$

If $p_n(x) = \sum_{r=1}^{n} c_{n,r}x^r$, then

$$p_n^{(r)} = D^r \sum_{i=1}^{n} c_{n,i}x^i = \sum_{i=r}^{n} c_{n,i} \frac{i!}{(i-r)!} x^{i-r} \tag{5.6}$$

Therefore, Equation (5.4) becomes

$$Qp_n = \sum_{r=1}^{n} \frac{1}{r!} \alpha_r (c_{n,r}r! + c_{n,r+1}\frac{(r+1)!}{1!}x + c_{n,r+2}\frac{(r+2)!}{2!}x^2 + \cdots + c_{n,n}\frac{n!}{(n-r)!}x^{n-r}) \tag{5.7}$$

And

$$np_{n-1} = n\{c_{n-1,1}\ x + c_{n-1,2}\ x^2 + c_{n-1,3}\ x^3 + \cdots + c_{n-1,n-1}\ x^{n-1}\} \tag{5.8}$$

By equating the constant terms from Equations (5.7) and (5.8), we get

$$\sum_{i=1}^{n} c_{n,i}\ \alpha_i = 0,\ \textit{whenever}\ n > 1$$

By equating the coefficients of $x^i, (i = 1, 2, 3...)$, from Equations (5.7) and (5.8), we get

$$\sum_{r=1}^{n} \alpha_r\ c_{n,r+1}\ \frac{(r+1)}{1!} = n\ c_{n-1,1}$$

$$\sum_{r=1}^{n} \alpha_r\ c_{n,r+2}\ \frac{(r+1)(r+2)}{2!} = n\ c_{n-1,2}$$

$$\sum_{r=1}^{n} \alpha_r\ c_{n,r+3}\ \frac{(r+1)(r+2)(r+3)}{3!} = n\ c_{n-1,3}$$

and so on.

By consolidating the above equations, we get

$$\sum_{r=1}^{n} \binom{r+i}{i} \alpha_r\ c_{n,r+i} = n\ c_{n-1,i}\quad (i = 1, 2, \cdots n).$$

Therefore, we obtain the following Theorem.

Theorem 6.

$p_n(x) = \sum_{r=1}^{n} c_{n,r}\ x^r$, where $n \geqslant 1$ is a basic polynomials sequence for some delta operator Q if and only if the coefficients of $p_n(x)$ satisfy the following identities.

1) $c_{1,1}\ \alpha_1 = 1$;
2) $\sum_{i=1}^{n} c_{n,i}\ \alpha_i = 0,\ \textit{whenever}\ n > 1$;
3) $\sum_{r=1}^{n} \binom{r+i}{i} \alpha_r\ c_{n,r+i} = n\ c_{n-1,i}\quad (i = 1, 2, \cdots n)$

Hence we have reintroduced the theorem stated as above, in terms of three new identities instead of binomial type characterization. According to G.C.Rota [1], the Theorem (3) gives a necessary and sufficient conditions for the basic polynomials in terms of binomial characterization. The same necessary and sufficient conditions for the basic polynomials corresponding to some delta operator Q is reconstructed in terms of the sequential characterization in the Theorem (6). Practical difficulty to fix some sequence of basic polynomials is reduced due to this Theorem (6).

6 Examples and Applications

In previous section 3, we successfully developed a method to chart out the sequence of basic polynomials and to find the corresponding delta operator. This method is based on Theorem (5) in section 2. Now we discuss the same theme through the Theorem(6). This theorem also gives a method to chart out the sequence of basic polynomials and to find the corresponding delta operator. We take the same special polynomials to discuss this method.

Example 1.

Taking $p_n(x) = x^n$.
Clearly $p_0(x) = 1$ *and* $p_1(x) = x \Rightarrow c_{1,1} = 1$

From Equation (1) in Theorem (6),

$$c_{1,1}\alpha_1 = 1 \Rightarrow \alpha_1 = 1$$

By Equation (2) in Theorem (6) and $c_{n,r} = \delta_{n,r}$, we have

$$\sum_{r=1}^{n} \alpha_r \ c_{n,r} = \sum_{r=1}^{n} \alpha_r \ \delta_{n,r} \Rightarrow \alpha_n = 0, \ n > 1$$

Thus we can easily verified that the characterization of the delta operator for the Monomials $\{x^n, n = 0, 1, 2, \cdots n\}$ being $\alpha_1 = 1$ and $\alpha_r = 0, (r = 2, 3, 4, \cdots n)$.

Example 2. Taking $p_n(x) = [x]_n = \{x(x-1)(x-2)\cdots(x-n+1)\}$
$p_1(x) = x \Rightarrow c_{1,1} = 1$

From Equation (1) in Theorem (6),

$$c_{1,1}\alpha_1 = 1 \Rightarrow \alpha_1 = 1$$

Since $p_n(1) = 0$, and Equation (2) in Theorem (6), we have

$$\sum_{r=1}^{n} c_{n,r} = p_n(1) = 0 = \sum_{r=1}^{n} \alpha_r \ c_{n,r}$$

Equating the corresponding terms , we get $\alpha_r = 1, \ r > 1$.

Thus we can verified that the set of polynomials $[x]_n$ is a sequence of basic polynomials and the characterization of the delta operator Q being $\alpha_n = 1$, for all $n \geqslant 1$.

Example 3.

Taking $p_n(x) = x(x-n)^{n-1}$
$p_1(x) = x \Rightarrow c_{1,1} = 1$

From Equation (1) in Theorem (6),

$$c_{1,1} \ \alpha_1 = 1 \Rightarrow \alpha_1 = 1$$

From Equation (2) in Theorem (6),

$$\sum_{r=1}^{n} \alpha_r \ c_{n,r} = 0, \ \textit{whenever} \ r \geq 2 \tag{6.1}$$

Since $p_n^{'}(1) = 0$, we have,

$$\sum_{r=1}^{n} r \ c_{n,r} = p_n^{'}(1) = 0 \tag{6.2}$$

Equating the corresponding terms from Equations (6.1) and (6.2), we get $\alpha_r = r$, whenever $r \geq 2$ Hence we can easily verified that the set of polynomials $p_n(x) = x(x-n)^{n-1}$ is a basic polynomials and the characterization of the delta operator Q being $\alpha_n = n$, for all $n \geqslant 1$.

Thus the Theorem (6) enriches the theory to decide whether a set of polynomials is a set of basic polynomials with respect to a definite delta operator numerically. But the earlier Theorem (3) does not have this characterization facility.

7 Status and Further Directions

Special polynomials play an important role in applicable analysis. Many of the models in Applied Mathematics are expressed and analyzed in terms of special polynomials. There are classically three approaches to define the special polynomials. They are generating functions, recursion relations and differential equations. It is envisaged that the finite operator calculus may be developed to fourth equivalent approach to define and analyze the special polynomials through the characterization of the delta operator. A.Maheswaran [5] discussed some properties of q_delta operators and their q_ basic polynomials. This is a good starting point for further investigation of the characterization of the q_delta operator for the q_ basic polynomials.

Competing Interests

Authors have declared that no competing interests exist.

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