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Some Remarks on Cusp Forms on the Full Modular Group Γ_1

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

We present some cusp forms and their Fourier coefficients on the full modular group Γ_1, using the adjoint linear maps, nonanalytic Poincare series and Hecke operators.

Keywords: Cusp forms; nonanalytic Poincare series; Dirichlet series of Rankin type; Hecke operators.

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1 Introduction

Let k be a positive integer and denote by S_k the space of cusp forms and by M_k the space of modular forms of weight k on the modular group Γ_1. We shall use H to denote the upper half plane, $\mathbb{C}$ for the set of complex numbers.

Let f and g be forms in M_k with Fourier coefficients $a(m)$ and $b(m)$ respectively. For a positive integer n define a Dirichlet series of Rankin type by

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$$L_{f,g;n}(s)=\sum_{m\geq 1}\frac{\overline{a(m+n)}b(m)}{(m+n)^s}$$

By Deligne's estimate, previously the Ramanujan-Petersson conjecture, $L_{f,g;n}(s)$ is absolutely convergent for Re(s)> $\frac{k+t}{2}$. It can be shown that $L_{f,g;n}(s)$ has a meromorphic continuation to $\mathbb{C}$.

Let $f, f' \in M_k$ such that f or f' is a cusp form. The Petersson scalar product is defined by

$$\langle f, f'\rangle=\int_K f(\tau)\overline{f'(\tau)}y^k dV$$

in [1]. Where $\tau = x+iy$, $dV=\frac{dxdy}{y^2}$ and K is a fundamental domain for the action of Γ_1 on H.

In [2, p. 115], nonanalytic Poincare series is defined by

$$G_\nu(\tau\,|\,z)=\sum_{(c,d)=1}\sum\frac{\exp\{2\pi i(\nu+\kappa)M\tau\}}{\upsilon(M)(c\tau+d)^k\left|c\tau+d\right|^z} \tag{1}$$

where $M=\begin{pmatrix} * & * \\ c & d\end{pmatrix}\in\Gamma_1$, $|\,c\tau+d\,|^z$ is the Hecke convergence factor, Im τ>0, ν is an arbitrary integer and υ is a multiplier system (MS) for Γ_1 in the weight k. The number κ is determined from υ by $\upsilon(S)=e^{2\pi i\kappa}, 0\leq\kappa<1, S=\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}$. Eventually z can be thought of as an arbitrary complex number, but in order to guarantee absolute convergence of the double series (1) we assume initially that Rez>2-k. Uniform convergence of the series of absolute values implies that $G_\nu(\tau\,|\,z)$ is holomorphic (in the variable z) in the half-plane Rez>2-k and, as a function of $\tau\in$ H, it satisfies the transformation formula

$$G_\nu(M\tau\,|\,z)=\upsilon(M)(c\tau+d)^k\ |\,c\tau+d\,|^z\ G_\nu(\tau\,|\,z) \tag{2}$$

for all $M=\begin{pmatrix} * & * \\ c & d\end{pmatrix}\in\Gamma_1$.

In [2, p.118], the Fourier expansion of $G_\nu(\tau\,|\,z)$ is given by

$$G_\nu(\tau\,|\,z)-2e^{2\pi i(\nu+\kappa)\tau}=2i^{-k}\frac{(2\pi)^{k+z}}{\Gamma(z/2)}\sum_{n=0}^{\infty}(n+\kappa)^{k+z-1}e^{2\pi i(n+\kappa)\tau}\sum_{p=0}^{\infty}\frac{\left\{-4\pi^2(n+\kappa)(\nu+\kappa)\right\}^p}{p!\Gamma(k+p+z/2)}$$

$$\times\sigma(4\pi(n+\kappa)y, k+p+z/2, z/2)Z_n(z/2+k/2+p)$$

$$+2i^{-k}\frac{(2\pi)^{k+z}}{\Gamma(z/2)}\sum_{n=0}^{\infty}(n-\kappa)^{k+z-1}e^{-2\pi i(n-\kappa)\bar{\tau}}\sum_{p=0}^{\infty}\frac{\{-4\pi^2(n-\kappa)(\nu+\kappa)\}^p}{p!\Gamma(k+p+z/2)}$$

$$\times\sigma(4\pi(n-\kappa)y, z/2, k+p+z/2)Z_{-n}(z/2+k/2+p) \qquad (3)$$

where $Z_n(w)=\sum_{c=1}^{\infty}A_{c,\nu}(n,\upsilon)c^{-2w}$ is Selberg's Kloosterman zeta-function and

$\sigma(\eta,\alpha,\beta)=\int_0^{\infty}(u+1)^{\alpha-1}u^{\beta-1}e^{-\eta u}du$ is the notation of Siegel.

In [2, p.125], the function $F_\nu(\tau\mid z)$ is defined by

$$F_\nu(\tau\mid z)=y^{z/2}G_\nu(\tau\mid z) \qquad (4)$$

as a function of τ and z. Where $\tau=x+iy$. It follows from (2) that, $F_\nu(\tau\mid z)$ satisfies the transformation formulae

$$F_\nu(M\tau\mid z)=\upsilon(M)(c\tau+d)^k F_\nu(\tau\mid z)$$

By the Fourier expansion (3) of $G_\nu(\tau\mid z)$ and from (4), we obtain the Fourier expansion of $F_\nu(\tau\mid z)$ at the cusp point ∞ of the form

$$F_\nu(\tau\mid z)=\sum_{n=0}^{\infty}a_1(n)e^{2\pi i(n+\kappa)\tau}+\sum_{n=0}^{\infty}a_2(n)e^{-2\pi i(n-\kappa)\bar{\tau}}$$

where the Fourier coefficients $a_1(n)$ and $a_2(n)$ depend upon z. Hence, $F_\nu(\tau\mid z)$ is a modular form of weight k and MS υ.

In [2, p. 125], the following lemma is given.

Lemma 1.1: Suppose $\nu+\kappa>0$, Rez>2-k and f(τ) is a cusp form of weight k and MS υ on Γ_1. Then,

$$\langle F_\nu, f\rangle = 2\bar{b}_\nu\Gamma(k-1+z/2)\{4\pi(\nu+\kappa)\}^{1-k-z/2}$$

where $f(\tau)=\sum_{n+\kappa>0}b_n e^{2\pi i(n+\kappa)\tau}$ and the bar denotes the conjugate complex number.

For $n=\nu+\kappa$, we shall write $F_{k-t,n}(\tau\mid z)$ instead of $F_\nu(\tau\mid z)$. Thus, we have

$$F_{k-t,n}(\tau\mid z)=y^{z/2}G_{k-t,n}(\tau\mid z)=y^{z/2}\sum_{(c,d)=1}\sum\frac{\exp\{(2\pi in)M\tau\}}{\upsilon(M)(c\tau+d)^{k-t}|c\tau+d|^z}$$

In [3], the Hecke operator T_n is defined on M_k by the equation

$$(T_n f)(\tau) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f(\frac{n\tau + bd}{d^2})$$

for a fixed integer k and any n=1,2,…

Theorem 1.2: If $f \in M_k$ and has the Fourier expansion

$$f(\tau) = \sum_{m=0}^{\infty} a(m) e^{2\pi i m \tau}$$

then $T_n f$ has the Fourier expansion

$$(T_n f)(\tau) = \sum_{m=0}^{\infty} a_n(m) e^{2\pi i m \tau}$$

where,

$$a_n(m) = \sum_{d|(n,m)} d^{k-1} a(\frac{nm}{d^2}) . \tag{5}$$

for n=1,2,…

In [3], Klein's modular function $J(\tau)$ is defined by

$$J(\tau) = \frac{g_2^3(\tau)}{\Delta(\tau)}$$

and analytic in H. Where $\Delta(\tau) = g_2^3(\tau) - 27 g_3^2(\tau)$, $g_2(\tau) = 60 \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^4}$ and $g_3(\tau) = 140 \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^6}$.

Theorem 1.3: If $\tau \in$ H, we have the Fourier expansion

$$j(\tau) = 12^3 J(\tau) = e^{-2\pi i \tau} + 744 + \sum_{n=1}^{\infty} c(n) e^{2\pi i n \tau}$$

where $c(n)$ are integers. [3]

In [4], W. Kohnen proved the following theorem using analytic Poincare series and the properties of inner product.

Theorem 1.4: The function

$$W_g(f)(z)=\sum_{n\geq 1} n^{k-t-1}L'_{f,g;n}(k-1)e^{2\pi i n z}\ ,\ (z\in \mathrm{H})$$

is a cusp form of weight $k-t$ on Γ_1. In fact, the map $W_g: S_k \to S_{k-t},\ f \mapsto c_{k,t}W_g(f)$

is the adjoint w.r.t. the usual Petersson scalar products of the map $S_{k-t} \to S_k, h \mapsto gh.$ Where

$$c_{k,t}:=\frac{\Gamma(k-1)}{\Gamma(k-t-1)(4\pi)^t}, L'_{f,g;n}(s)=\sum_{m\geq 1}\frac{a(m+n)\overline{b(m)}}{(m+n)^s} \tag{6}$$

In [5], Min Ho Lee obtained the Fourier coefficients of Siegel cusp form $\phi_g^* f$ in terms of Dirichlet series of Rankin type associated to the Fourier coefficients of Siegel cusp forms f and g.

In this paper, we shall obtain some cusp forms of integer weight on Γ_1, using nonanalytic Poincare series and the properties of inner product. Further, the Fourier coefficients of cusp form $P_g^* f$ of weight k-t on Γ_1 are written in terms of Dirichlet series of Rankin type associated to the Fourier coefficients of cusp forms f and g of weights k and t respectively. Using the properties of Hecke operators T_n, some results for the Fourier coefficients of cusp form $T_n f$ are also given.

For several recent results concerning Modular forms, we refer the reader to [6-11].

2 The Results

Theorem 2.1: Let t be a positive integer, k an integer with k>t+2. Let $f(\tau)\in S_k$ and $g(\tau)\in S_t$. Then the function, for $\tau\in \mathrm{H}$ and $Re z > 2-k$

$$U_g(f)(\tau)=\sum_{n\geq 1} a(P_g^* f,n)e^{2\pi i n\tau}$$

is a cusp form of weight k-t on Γ_1. Where

$$a(P_g^* f,n)=\frac{\Gamma(k-1)\overline{(4\pi)^{\frac{z}{2}-t}}}{2\Gamma(k-t-1+\frac{z}{2})}\overline{n^{k-t-1+\frac{z}{2}}}L_{f,g;n}(k-1) \tag{7}$$

Proof: Let t be a positive integer, k an integer with k>t+2. Let $f\in S_k$ and $g\in S_t$. The map $P_g:S_{k-t}\to S_k$, $h\to gh$ is a linear homomorphism of finite dimensional Hilbert spaces and has an adjoint $P_g^*:S_k\to S_{k-t}$. Let $a(P_g^* f,n)$ be the n^{th} Fourier coefficient of $P_g^* f$. Since $F_{k-t,n}$ is a modular form of weight k-t, by Petersson scalar product and using Lemma 1.1, we obtain

$$
\begin{aligned}
w_t 2.\overline{a(P_g^* f, n)} &= \langle P_g^* f, F_{k-t,n} \rangle \\
&= \langle f, P_g F_{k-t,n} \rangle \\
&= \langle f, gF_{k-t,n} \rangle \\
&= \int_K H(\tau) \overline{F_{k-t,n}} y^{k-t} dV
\end{aligned}
$$

where $H(\tau) = f(\tau)\overline{g(\tau)} y^t$, $w_t = \dfrac{\Gamma(k-t-1+\frac{z}{2})}{(4\pi n)^{k-t-1+\frac{z}{2}}}$. From the transformation formulas of f and g, the function H(τ) is a modular form of weight k-t on Γ_1. Hence, we write

$$2\overline{w_t} a(P_g^* f, n) = \int_0^\infty a(\overline{H}, n, y) e^{-2n\pi y} y^{k-t-2} dy$$

Thus, we have

$$a(P_g^* f, n) = \frac{1}{2\overline{w_t}} \int_0^\infty a(\overline{H}, n, y) e^{-2\pi n y} y^{k-t-2} dy$$

where $a(\overline{H}, n, y)$ is the n^th^ Fourier coefficient of $\overline{H(\tau)}$ w.r.t. the variable $e^{2\pi i x}$. Using the Fourier expansions of f and g in the definition of H, we obtain

$$a(\overline{H}, n, y) = y^t \sum_{m \geq 1} \overline{a(m+n)} b(m) e^{-2\pi(2m+n)y}$$

where a(m) and $b(m)$ are Fourier coefficients of functions f and g respectively.

Hence, by Mellin's transform, we find

$$a(P_g^* f, n) = \frac{\Gamma(k-1)\overline{(4\pi)^{\frac{z}{2}-t}}}{2\Gamma(k-t-1+\frac{z}{2})} \overline{n^{k-t-1+\frac{z}{2}}} \sum_{m \geq 1} \frac{\overline{a(m+n)} b(m)}{(m+n)^{k-1}}$$

where $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$, Re(s)>0. This concludes the proof.

Theorem 2.2: Let k be an integer with k>2. Let $g_1(\tau)$ be a modular function with respect to Γ_1 which is analytic on H and $f(\tau) \in S_k$. Then the function, for $\tau \in$ H and $Rez > 2 - k$,

$$U_{g_1}(f)(\tau)=\sum_{n\geq 1}a(T_p f,n)e^{2\pi i n\tau}$$

is a cusp form of weight k on Γ_1. Where

$$a(T_p f,n)=\frac{\overline{\Gamma(k-1)(4\pi)^{\frac{z}{2}}}}{\overline{2\Gamma(k-1+\frac{z}{2})}}\overline{n^{k-1+\frac{z}{2}}}L_{f,g_1;n}(k-1) \tag{8}$$

for $n=p$ and $m,p=1,2,...$

Proof: Let $f(\tau)\in S_k$. Let k be an integer with k>2. Let $T_p:S_k\to S_k$ be a Hecke operator such that $h\to g_1 h$. Where $g_1(\tau)$ is a modular function with respect to Γ_1 which is analytic on H. Since the Hecke operators are Hermitian on S_k, using Lemma 1.1 and from Petersson scalar product, we obtain

$$\begin{aligned} w_0 2.\overline{a(T_p f,n)} &= \langle T_p f, F_{k,n}\rangle \\ &= \langle f, T_p F_{k,n}\rangle \\ &= \langle f, g_1 F_{k,n}\rangle \\ &= \int_K H_1(\tau)\overline{F_{k,n}}y^k dV \end{aligned}$$

where $H_1(\tau)=f(\tau)\overline{g_1(\tau)}$, $w_0=\dfrac{\Gamma(k-1+\frac{z}{2})}{(4\pi n)^{k-1+\frac{z}{2}}}$ and $H_1(\tau)\in M_k$.

Hence, we get

$$2a(T_p f,n)=\frac{1}{\overline{w_0}}\int_0^\infty a(\overline{H_1},n)e^{-2\pi n y}y^{k-2}dy$$

where $a(\overline{H_1},n)$ is the n[th] Fourier coefficient of $\overline{H_1(\tau)}$ w.r.t. the variable $e^{2\pi i x}$. From the Fourier expansions of $f(\tau)$ and $g_1(\tau)$, we have

$$a(\overline{H_1},n)=\sum_{m\geq 1}\overline{a(m+n)}b_1(m)e^{-2\pi(2m+n)y}$$

where a(m) and $b_1(m)$ are Fourier coefficients of $f(\tau)$ and $g_1(\tau)$ respectively.

By Mellin's transform, we find,

$$a(T_p f,n)=\frac{\overline{\Gamma(k-1)(4\pi)^{\frac{z}{2}}}}{2\Gamma(k-1+\frac{z}{2})}\,\overline{n^{k-1+\frac{z}{2}}}\sum_{m\geq 1}\frac{\overline{a(m+n)}b_1(m)}{(m+n)^{k-1}}$$

and by (5)

$$a_p(m)=a(T_p f,n)=\frac{\overline{\Gamma(k-1)(4\pi)^{\frac{z}{2}}}}{2\Gamma(k-1+\frac{z}{2})}\,\overline{n^{k-1+\frac{z}{2}}}\;L_{f,g_1;n}(k-1)$$

for $n=p$ and $m,p=1,2,...$ This completes the proof.

The proof of the following Theorem is similar to that of the Theorem 2.2 and by using Theorem 1.4

Theorem 2.3: Let k be an integer with k>2. Let $g_1(\tau)$ be a modular function with respect to Γ_1 which is analytic on H and $f(\tau)\in S_k$. Then the function

$$W_{g_1}(f)(\tau)=\sum_{n\geq 1}a(T_p f,n)e^{2\pi i n\tau}$$

is a cusp form of weight k on Γ_1. Where

$$a(T_p f,n)=n^{k-1}L'_{f,g_1;n}(k-1) \tag{9}$$

for $n=p$ and $m,p=1,2,...$ and $L'_{f,g_1,n}(k-1)$ is given by (6).

3 Numerical Examples

Example 1. Let $f(\tau)=\ \Delta(\tau)=g_2^3(\tau)-27g_3^2(\tau)$ and $g_1(\tau)=1728J(\tau)$. Since the discriminant function $\Delta(\tau)$ is a cusp form of weight 12 and from (8) and (9), we have

$$a_p(m)=a(T_p\Delta,n)=\frac{\overline{10!(4\pi)^{\frac{z}{2}}}}{2\Gamma(11+\frac{z}{2})}\,\overline{n^{11+\frac{z}{2}}}\sum_{m\geq 1}\frac{\overline{\tau(m+n)}c(m)}{(m+n)^{11}}.$$

and

$$a_p(m)=a(T_p\Delta,n)=n^{11}\sum_{m\geq 1}\frac{\tau(m+n)\overline{c(m)}}{(m+n)^{11}}$$

as the Fourier coefficients of $T_n\Delta$, respectively. Where $c(n)$ are the Fourier coefficients of Klein's j-invariant and $\tau(m)$ is Ramanujan's tau function.

Example 2. Let $f(\tau) \in S_k$, $k > 14$ and $g(\tau) = \Delta(\tau)$ the discriminant function. From (7), we obtain

$$a(P_g^* f, n) = \frac{\Gamma(k-1)\overline{(4\pi)^{\frac{z}{2}-12}}}{2\overline{\Gamma(k-13+\frac{z}{2})}} n^{\overline{k-13+\frac{z}{2}}} \sum_{m\geq 1} \frac{\overline{a(m+n)}\tau(m)}{(m+n)^{k-1}}$$

as the Fourier coefficiens of $P_g^* f$. Where $\tau(m)$ is Ramanujan's tau function.

4 Conclusion

Some cusp forms on the full modular group and their Fourier coefficients are obtained. Therefore, the result of W. Kohnen's paper [4], from Poincare series to a nonanalytic Poincare series, is extended.

Competing Interests

Author has declared that no competing interests exist.

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