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Finite Time Blow-up, Extinction and Non-extinction of Solutions for an Evolutionary Problem

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Authors' contributions

This work was carried out in collaboration between both authors. Author ZY designed the study and guided the research. Author ZZ performed the analysis and wrote the first draft of the manuscript. Authors ZY and ZZ managed the analyses of the study. Both authors read and approved the final manuscript.

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Abstract

In this paper we consider a class of p-biharmonic parabolic equation with nonlocal nonlinearities and Neumann boundary condition. By constructing suitable auxiliary functions and using differential inequalities, we give blow-up criterion of solutions as well as extinction and non-extinction. In addition, we derive similar results for a different equation.

Keywords: p-biharmonic parabolic equation; blow-up; extinction;

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1 Introduction

During the past few decades, the development of evolutionary problems in biology, ecology and biochemistry, and the traditional importance of these systems in physics, heat-mass transfer lead to extensive study in various aspects of nonlinear parabolic partial differential equations. A special topic of the analysis is the finite time blow-up phenomena of solutions. The papers referenced here have been investigated in the questions of existence and nonexistence of global solutions, blow-up of solutions, blow-up rates and bounds on blow-up time, and asymptotic behavior of solutions to semilinear and nonlinear problems. In this work we consider the following nonlocal initial boundary value problem,

$$\begin{cases} u_t - \gamma\Delta u_t + \Delta(|\Delta u|^{p-2}\Delta u) = f(|u|) - \frac{1}{m(\Omega)}\int_\Omega f(|u|)dx & (x,t) \in \Omega \times (0,t^*), \\ |\Delta u|^{p-2}\frac{\partial u}{\partial \nu} = |\Delta u|^{p-2}\frac{\partial \Delta u}{\partial \nu} = 0 & (x,t) \in \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) & x \in \bar{\Omega}, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R^N} (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $f : [0,\infty) \mapsto [0,\infty)$ is a locally Lipschitz function, $m(\Omega)$ represents the Lebesgue measure of the domain Ω, $\gamma \geq 0$, $\max[1, \frac{2N}{N+4}] < p \leq 2$ such that $W^{2,p}(\Omega) \hookrightarrow L^2(\Omega)$, $u_0(x) \in L^\infty(\Omega) \cap W^{2,p}(\Omega)$, $\int_\Omega u_0(x)dx = 0$.

The particular case where $\gamma = 0$ and f is a power function of the form $f(u) = u^q$, with $0 < q \leq 1$ was recently considered in [1]. In fact, the authors in [1] considered the following p-biharmonic parabolic equation with nonlocal source

$$\begin{cases} u_t + \Delta(|\Delta u|^{p-2}\Delta u) = |u|^q - \frac{1}{m(\Omega)}\int_\Omega |u|^q dx & (x,t) \in \Omega \times (0,t^*), \\ |\Delta u|^{p-2}\frac{\partial u}{\partial \nu} = |\Delta u|^{p-2}\frac{\partial \Delta u}{\partial \nu} = 0 & (x,t) \in \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) & x \in \bar{\Omega}, \end{cases} \tag{1.2}$$

where Ω is a bounded domain of $\mathbf{R^N} (N \geq 1)$ with smooth boundary $\partial\Omega$, $m(\Omega)$ represents the Lebesgue measure of the domain Ω, $\max[1, \frac{2N}{N+4}] < p \leq 2$ such that $W^{2,p}(\Omega) \hookrightarrow L^2(\Omega)$, $u_0(x) \in L^\infty(\Omega) \cap W^{2,p}(\Omega)$, $\int_\Omega u_0(x)dx = 0$. They established the blowup, extinction and non-extinction results for the solutions to (1.1).

Moreover, a thin film equation with similar nonlocal reaction term was considered by authors in [2]

$$\begin{cases} u_t + u_{xxxx} = |u|^{p-1}u - \frac{1}{m(\Omega)}\int_\Omega |u|^{p-1}u dx & (x,t) \in \Omega \times (0,t^*), \\ u_x = u_{xxx} = 0 & (x,t) \in \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) & x \in \bar{\Omega}, \end{cases} \tag{1.3}$$

where $\Omega = (0,a), p > 1$, and $u_0 \in H^2(\Omega)$, with $u_0 \neq 0$. By potential well method they obtain a threshold result of global existence and non-existence for the sign-changing weak solutions.

Also, an evolutionary problem with diffusion term involving p-Laplacian operator was considered in [3]. The authors in [3] considered the following evolutionary problem

$$\begin{cases} u_t + \Delta_p u = f(|u|) - \frac{1}{m(\Omega)}\int_\Omega f(|u|)dx & (x,t) \in \Omega \times (0,t^*), \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} = 0 & (x,t) \in \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) & x \in \bar{\Omega}, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbf{R^N}$ is a bounded regular domain, and Δp, for $p \geq 2$, is the p-Laplacian operator. They studied the generalized convex functions and established the energetic criterion for blow-up solutions to (1.3). Further studies of generalized convexity can be seen in [4],[5].

Furthermore, in [6], a semilinear pseudo-parabolic equation was considered as follows

$$\begin{cases} u_t - \Delta u - \Delta u_t = u^p & (x,t) \in \Omega \times (0,t^*), \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) \geq 0 & x \in \bar{\Omega}, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbf{R}^{\mathbf{N}}$ is a bounded domain with smooth boundary, $u_0(x) \in H_0^1(\Omega)$. By means of differential inequality technique, they establish a blow up criterion and obtain bounds for blow up time under appropriate conditions. A more general equation was considered in [7] and further results were obtained.

The problems of type (1.1) arise naturally in mechanics, biology, and population dynamics. See [8]-[14]. For example, if we consider a couple or a mixture of two equations of the above type, the resulting problem describes the temperatures of two substances which constitute a combustible mixture, or represents a model for the behavior of densities of two diffusion biological species which interact each other. On the other hand, the blow-up phenomena of evolutionary problems with nonlocal source $|u|^q - \frac{1}{m(\Omega)} \int |u|^q dx$ were studied in a lot of papers, as appeared in [15],[16],[17].

As in [1], we consider the weak solutions as follows:

Definition 1.1 A function $u(x,t) \in L^\infty(0,T;L^\infty(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega))$ is said to be a weak solution to the problem (1.1), if $u_t \in L^2(0,T;L^2(\Omega)$ and for any $\varphi \in L(0,T,;W^{2,p}(\Omega))$ with $\frac{\partial \varphi}{\partial \nu}|_{\partial\Omega} = 0$, there holds

$$\int_0^T \int_\Omega [u_t\varphi + \nabla\varphi\nabla u_t + |\Delta u|^{p-2}\Delta u\Delta\varphi - (f(|u|) - \frac{1}{m(\Omega)} \int_\Omega f(|u|)dx)\varphi]dxdt = 0 \tag{1.6}$$

The local existence of the weak solution can be obtained by using Galerkin approximation method. Let $u(x,t)$ be the weak solution to the problem (1.1).

Motivated by the above works, we intend to study the blow-up phenomena and quenching behavior of the solution to problem (1.1). Here we take $\gamma = 1$ for simplicity. Similar results can be obtained for any positive γ. In detail, the paper is organized as follows: in Section 2, we derive conditions on the data of problem (1.1) sufficient to instigate the blow up of $u(x,t)$. In Section 3, we derive conditions on the data of problem (1.1) sufficient to ensure the non-extinction and extinction property of $u(x,t)$. In Section 4, we consider another type of p-biharmonic equation and derive the corresponding criterions which differ from results of above equation.

2 The Blow-up Solution

In this section, we determine a condition sufficient to ensure the solution blows up at finite time.

According to Hermite-Hadamard inequality, the mean value of a continuous convex function f : $[a,b] \to R$ lies between the value of f at the midpoint of the interval $[a,b]$ and the arithmetic mean of the values of f at the endpoints of this interval, that is,

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{HH}$$

Moreover, each side of this double inequality characterizes convexity in the sense that a real-valued continuous function f defined on an interval I is convex if its restriction to each compact subinterval $[a,b] \subset I$ verifies the left hand side of (HH)(equivalently, the right hand side of (HH)).

In what follows we will be interested in a class of generalized convex functions defined in [3] motivated by the right hand side of the Hermite-Hadamard inequality.

Definition 2.1. A real-valued function f defined on an interval $[a,\infty)$ belongs to the class GC_α(for some $\alpha > 0$), if it is continuous, nonnegative, and

$$\frac{1}{t-a}\int_a^t f(x)dx \leq \frac{1}{\alpha+1}f(t) \ \ for\ t\ large\ enough. \tag{2.1}$$

It is not difficult to see that by simple calculation the condition (2.1) is equivalent to the fact that the ratio

$$\frac{\frac{1}{t-a}\int_a^t f(x)dx}{(t-a)^\alpha} \tag{2.2}$$

is nondecreasing for t bigger than a suitable value $A \geq a$, which implies that the mean value $\frac{1}{t-a}\int_a^t f(x)dx$ has a polynomial growth at infinity.

Here we define auxiliary functions

$$E(u(t)) = \int_\Omega (\frac{1}{p}|\Delta u|^p - \int_0^u f(|\tau|)d\tau)dx, h(t) = \frac{1}{2}\int_\Omega u^2 + |\nabla u|^2 dx, H(t) = \int_0^t h(s)ds \tag{2.3}$$

where u is the solution of (1.1). We first start by noticing that each solution of the problem above has the property

$$\int_\Omega u dx = 0$$

because the integral in the right hand side of the first equation in (1.1) is 0 and

$$\begin{aligned}\frac{d}{dt}(\int_\Omega u dx) &= \int_\Omega u_t dx \\ &= \int_\Omega -\Delta(|\Delta u|^{p-2}\Delta u) + \Delta u_t dx = 0\end{aligned}$$

Hence, by the initial condition $\int_\Omega u_0 dx = 0$, we have $\int_\Omega u dx = 0$.

Throughout this paper, the norm of $L^r(\Omega)$ is denoted by $\| \cdot \|_r$. Since $W^{2,p}(\Omega) \hookrightarrow L^2(\Omega)$ and $\int_\Omega u dx = 0$. The optimal embedding constant B exists such that

$$\| u \|_2 \leq B \| \Delta u \|_p . \tag{2.4}$$

Furthermore, it is not difficult to obtain the following equality

$$\int_0^t \int_\Omega |u_s|^2 + |\nabla u_s|^2 dxds + E(t) = E(0) \tag{2.5}$$

According to this formula, if the initial energy $E(u_0)$ is non-positive, then $E(u(t))$ is non-positive for all $t > 0$. In the case of generalized convex functions of order α, we have

$$C\int_\Omega uf(|\ u\ |)dx \geq \int_\Omega \int_0^u f(|\ t\ |)dtdx \geq \frac{1}{p}\int_\Omega |\ \Delta u\ |^p\ dx. \tag{2.6}$$

where $C = \frac{1}{1+\alpha}$.

The main result of this section is the following theorem.

Theorem 2.1. Let u be the solution of problem (1.1). Assume that $f : [0,\infty) \mapsto [0,\infty)$ is a locally Lipschitz function belonging to the class GC_α, with $p < 1+\alpha$, and let u be a solution of the problem (1.1) corresponding to an initial data $u_0 \in C(\bar{\Omega})$, u_0 not identically zero. If $E(u_0) \leq 0$, then there is $T > 0$ such that

$$\limsup_{t\to T^-} h(t) = \infty \tag{2.7}$$

Notice that the condition $E(u_0) \leq 0$ in Theorem 2.1 is also necessary for the blow-up in finite time. In fact, (2.5) forces that

$$\inf[E(u(t)) : 0 < t < T] = -\infty.$$

This can be argued by contradiction. If $E(u(t)) \geq -C_0$, for some $C_0 > 0$, then we have

$$h'(t) = \int_\Omega uu_t + |\nabla u||\nabla u_t|dx \leq \frac{1}{2}\int_\Omega (u^2 + u_t^2 + |\nabla u|^2 + |\nabla u_t|^2)dx \leq (h(t) - E'(u(t)))$$

which yields

$$(h(t) + E(u(t)) + C_0)' \leq h(t) \leq h(t) + E(u(t)) + C_0.$$

Therefore,

$$h(t) \leq h(t) + E(u(t)) + C_0 \leq (h(0) + E(u_0) + C_0)e^t, \quad for\ all\ t \in (0, T).$$

and thus $h(t)$ is bounded.

The proof of Theorem 2.1 needs a preparation.

Lemma 1. Under the assumptions of Theorem 2.1, with $C = \frac{1}{\alpha+1}$, the two auxiliary functions $h(t)$ and $H(t)$ verify the following two conditions:

$$h'(t) \geq \frac{1}{C}\int_0^t \int_\Omega u_t^2 + |\nabla u_t|^2 dxdt; \tag{2.8}$$

$$\frac{1}{2C}(H'(t) - H'(0))^2 \leq H(t)H''(t). \tag{2.9}$$

Proof. By differentiation and the generalized convexity (2.6), we obtain

$$\begin{aligned} h'(t) &= \int_\Omega uu_t + |\nabla u||\nabla u_t|dx \\ &= \int_\Omega u(-\Delta(|\Delta u|^{p-2}\Delta u) + f(|u|))dx \\ &= \int_\Omega -|\Delta u|^p + uf(|u|)dx \\ &\geq \int_\Omega (-|\Delta u|^p + \frac{1}{C}\int_0^u f(t)dt)dx \\ &= -\frac{1}{C}\int_\Omega (\frac{1}{p}|\Delta u|^p - \int_0^u f(|\tau|)d\tau)dx + (\frac{1}{Cp} - 1)\int_\Omega |\Delta u|^p dx. \end{aligned} \tag{2.10}$$

Hence,

$$\begin{aligned} h'(t) &\geq -\frac{1}{C}E(u) + (\frac{1}{Cp} - 1)\int_\Omega |\Delta u|^p dx \\ &\geq -\frac{1}{C}E(u) \\ &= -\frac{1}{C}E(u_0) + \frac{1}{C}\int_0^t \int_\Omega u_t^2 + |\nabla u_t|^2 dxdt \\ &\geq \frac{1}{C}\int_0^t \int_\Omega u_t^2 + |\nabla u_t|^2 dxdt. \end{aligned}$$

Since

$$\begin{aligned} H'(t) - H'(0) &= \int_0^t h'(s)ds = \int_0^t \int_\Omega uu_t + |\nabla u||\nabla u_t| dxdt \\ &\leq (\int_0^t \int_\Omega u^2 dxdt)^{1/2}(\int_0^t \int_\Omega u_t^2 dxdt)^{1/2} + (\int_0^t \int_\Omega |\nabla u|^2 dxdt)^{1/2}(\int_0^t \int_\Omega |\nabla u_t|^2 dxdt)^{1/2} \\ &\leq (H(t))^{1/2}(2Ch'(t))^{1/2} = (2CH(t)H''(t))^{1/2}, \end{aligned}$$

by (2.10) we infer that

$$H'(t) - H'(0) = \int_o^t h'(s)ds \geq 0,$$

and thus

$$\frac{1}{2C}(H'(t) - H'(0))^2 \leq H(t)H''(t). \quad \square$$

Proof of Theorem 2.1. Assume the contrary, that the solution $u(x,t)$ exists for all $T > 0$. For any $t_0 > 0$, we claim that

$$\int_0^{t_0} \int_\Omega |u_s|^2 + |\nabla u_s|^2 dxds > 0. \tag{2.11}$$

Otherwise, there exists a $t_0 > 0$ such that $\int_0^{t_0} \int_\Omega |u_s|^2 + |\nabla u_s|^2 dxds = 0$, and hence $u_t = \nabla u_t = 0$ for a.e.$(x,t) \in \Omega \times (0,t_0)$. Then it follows from (2.10) that $\int_\Omega |\Delta u|^p dx = \int_\Omega uf(|u|)dx$ for a.e. $t \in (0,t_0)$, thus we get from (2.5) that

$$E(t) \geq (\frac{1}{p} - C)\int_\Omega |\Delta u|^p dx$$

for a.e. $t \in (0,t_0)$, which combines $E(t) \leq E(0) \leq 0$ and $p < 1 + \alpha$ imply $\int_\Omega |\Delta u|^p dx = 0$ for a.e. $t \in (0,t_0)$. By (2.4), we have $\|u(\cdot,t)\|_2 = 0$ for a.e. $t \in (0,t_0)$. Furthermore, since $u \in C([0,t_0], L^2(\Omega))$, we obtain $\|u(\cdot,t)\|_2 = 0$ for all $t \in [0,t_0]$, especially $\|u_0\|_2 = 0$, which contradicts to the assumption $\|u_0\| > 0$. Then (2.11) follows.

Fix $t_0 > 0$, and let $\delta = \int_0^{t_0} \int_\Omega |u_s|^2 + |\nabla u_s|^2 dxds$. By (2.11), we know that δ is a positive constant. Integrating (2.8) over (t_0,t), we obtain

$$\begin{aligned} h(t) &\geq h(t_0) + \frac{1}{C}\int_{t_0}^t \int_0^s \int_\Omega |u_\tau|^2 + |\nabla u_\tau|^2 dxd\tau ds \\ &\geq \int_{t_0}^t \int_0^s \int_\Omega |u_\tau|^2 + |\nabla u_\tau|^2 dxd\tau ds \geq \delta(t - t_0). \end{aligned}$$

Hence,

$$\lim_{t\to\infty} H'(t) = \lim_{t\to\infty} h(t) = +\infty.$$

which yields, for each $\beta \in (0, \frac{1}{C})$, the existence of a number $T_0 > 0$ such that for all $t > T_0$,

$$\beta H'(t)^2 \leq \frac{1}{C}(H'(t) - H'(0))^2.$$

Now, by (2.9) we obtain

$$\beta H'(t)^2 \leq 2H(t)H''(t).$$

We will show, by considering the function $G(t) = H(t)^{-q}$, for a suitable $q > 0$, that the last inequality leads to a contradiction. In fact,

$$\begin{aligned} G''(t) &= qH(t)^{-q-2}((q+1)(H'(t))^2 - H(t)H''(t)) \\ &\leq qH(t)^{-q-2}(\frac{2(q+1)}{\beta} - 1)H(t)H''(t) \end{aligned}$$

for all $t > T_0$, so that for $\beta \in (0, 1/C)$ and $q \in (0, 1/(2C) - 1)$ with $2(q+1) < \beta < 1/C$, the corresponding function G(t) is concave.

By (2.9), $\lim_{t\to\infty} H(t) = \infty$, whence $\lim_{t\to\infty} G(t) = 0$. Thus G provides an example of a concave and strictly positive function which tends to 0 at infinity, a fact which is not possible. Consequently, u may blow up at some finite time T. Thus completes the proof.

3 Non-extinction and Extinction Criterion

First, we determine sufficient conditions to ensure the solution to problem (1.1) does not extinct. The main result of this section is as follows.

Theorem 3.1. Let u be the solution of problem (1.1). Assume that $f : [0, \infty) \mapsto [0, \infty)$ is a locally Lipschitz function belonging to the class GC_α, with $p > 1+\alpha$, and let u be a solution of the problem (1.1) corresponding to an initial data $u_0 \in C(\bar{\Omega})$, u_0 not identically zero. If $E(u_0) \leq 0$, then the solution to problem(1.1) does not extinct in finite time.
The proof of Theorem 3.1 needs a preparation.

Lemma 3.1. ([15], Lemma 1.2) Suppose that $\theta > 0$, $a > 0$, $b > 0$ and $h(t)$ is a nonnegative and absolutely continuous function satisfying $h'(t) + ah^\theta(t) \geq b$, then for $0 < t < \infty$, it holds

$$h(t) \geq \min[h(0), (\frac{b}{a})^{\frac{1}{\theta}}].$$

Proof of Theorem 3.1. By differentiation and the generalized convexity (2.6), we obtain

$$\begin{aligned} h'(t) &= -\int_\Omega |\Delta u|^p + \int_\Omega uf(u)dx \\ &= -pE(u) - p\int_\Omega \int_0^u f(|\tau|)d\tau dx + \int_\Omega uf(|u|)dx \\ &\geq -pE(u_0) + (1-pC)\int_\Omega uf(|u|)dx \\ &\geq -pE(u_0) + (1-pC)(\int_\Omega u^2)^{\frac{1}{2}}(\int_\Omega f^2dx)^{\frac{1}{2}} \\ &\geq -pE(u_0) + Ah^{\frac{1}{2}}(t) \end{aligned} \tag{3.1}$$

where $A = (1-pC)(\int_\Omega f^2dx)^{\frac{1}{2}}$.

Therefore, by Lemma 3.1 and $E(u_0) < 0$, we have

$$h(t) \geq \min[h(0), (\frac{-pE(u_0)}{A})^2], \ \ t > 0.$$

Since $h(0) > 0, a > 0, E(u_0) < 0$, we get $h(t) > 0$ for all $t > 0$. $\square$

Then we derive certain conditions to ensure the quenching behavior of the solution to (1.1) in a particular case.

Lemma 3.2. Assume $0 < l < r \leq 1, \alpha \geq 0, \beta \geq 0$ and $\varphi(t)$ is a nonnegative and absolutely continuous function, which satisfies

$$\begin{cases} \varphi'(t) + \alpha\varphi^l(t) \leq \beta\varphi^r(t), & t \geq 0, \\ \varphi(0) > 0, \beta\varphi^{r-1}(0) < \alpha, \end{cases} \tag{3.2}$$

then it holds

$$\begin{cases} \varphi(t) \leq [-\alpha_0(1-l)t + \varphi^{1-l}(0)]^{\frac{1}{1-l}}, & 0 < t < T_0, \\ \varphi(t) \equiv 0, \qquad t \geq T_0 \end{cases} \tag{3.3}$$

where $\alpha_0 = \alpha - \beta\varphi^{r-l}(0) > 0$ and $T_0 = \alpha_0^{-1}(1-l)^{-1}\varphi^{1-l}(0)$.

Theorem 3.2. Let $u(x,t)$ be the solution of (1.1). Assume the following conditions on f and p, q, γ:

$$f(s) \leq \kappa s^q, \ \ \kappa > 0, \ \ p-1 < q \leq 1, \ \ \gamma = 0 \tag{3.4}$$

Moreover, we assume the initial data satisfy the condition

$$0 < \kappa \parallel u_0 \parallel_2^{q+1-p} < B^{-p}|\Omega|^{\frac{q-1}{2}}$$

Then the solution to (1.1) quenches in finite time. Furthermore, we have the following estimates:

$$\begin{cases} \|u(t)\|_2 \leq [\|u_0\|_2^{2-p} - (2-p)(B^{-p} - \kappa|\Omega|^{\frac{1-q}{2}}\|u_0\|_2^{q+1-p})t]^{\frac{1}{2-p}}, & 0 < t < T_* \\ \|u\|_2 = 0, \qquad t \geq T_* \end{cases} \tag{3.5}$$

where $T_* = [(2-p)(B^{-p} - \kappa \mid \Omega \mid^{\frac{1-q}{2}} \|u_0\|_2^{q+1-p})]^{-1}\|u_0\|_2^{2-p}$.

Proof of Theorem 3.2. Multiply the first equation of (1.1) by u and integrate over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega |\Delta u|^p dx = \int_\Omega u f(|u|)dx.$$

Define $\varphi(t) = \frac{1}{2}\int_\Omega u^2 dx$, then the above equation is equivalent to the following inequality

$$\varphi'(t) + \int_\Omega |\Delta u|^p dx = \int_\Omega u f(|u|)dx.$$

By (2.4), (3.4) and Holder's inequality we have

$$\varphi'(t) + 2^{\frac{p}{2}} B^{-p} \varphi^{\frac{p}{2}}(t) \le 2^{\frac{q+1}{2}} \kappa |\Omega|^{\frac{1-q}{2}} \varphi^{\frac{q+1}{2}}(t). \tag{3.6}$$

Then the conclusion follows by $\|u(\cdot,t)\|_2 = \sqrt{2\varphi(t)}$ and Lemma 3.2. $\square$

4 Another Case

In this section, we consider a different kind of p-biharmonic parabolic equation as follows,

$$\begin{cases} u_t - \gamma\Delta u_t - \Delta(|\Delta u|^{p-2}\Delta u) = f(|u|) - \frac{1}{m(\Omega)}\int_\Omega f(|u|)dx & (x,t) \in \Omega \times (0,t^*), \\ |\Delta u|^{p-2}\frac{\partial u}{\partial \nu} = |\Delta u|^{p-2}\frac{\partial \Delta u}{\partial \nu} = 0 & (x,t) \in \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) \qquad x \in \bar{\Omega}, \end{cases} \tag{4.1}$$

where $\Omega \subset bfR^N (N \ge 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $f : [0,\infty) \mapsto [0,\infty)$ is a locally Lipschitz function, $m(\Omega)$ represents the Lebesgue measure of the domain Ω, $\gamma \ge 0$, $\max[1, \frac{2N}{N+4}] < p \le 2$ such that $W^{2,p}(\Omega) \hookrightarrow L^2(\Omega)$, $u_0(x) \in L^\infty(\Omega) \cap W^{2,p}(\Omega)$, $\int_\Omega u_0(x)dx = 0$.

We first consider the weak solutions as follows:

Definition 4.1. A function $u(x,t) \in L^\infty(0,T;L^\infty(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega))$ is said to be a weak solution to the problem (1.1), if $u_t \in L^2(0,T;L^2(\Omega)$ and for any $\varphi \in L(0,T,;W^{2,p}(\Omega))$ with $\frac{\partial \varphi}{\partial \nu}|_{\partial\Omega} = 0$, there holds

$$\int_0^T \int_\Omega [u_t\varphi + \nabla\varphi\nabla u_t - |\Delta u|^{p-2}\Delta u \Delta\varphi - (f(|u|) - \frac{1}{m(\Omega)}\int_\Omega f(|u|)dx)\varphi]dxdt = 0 \tag{4.2}$$

The local existence of the weak solution can be obtained by using Galerkin approximation method. Let $u(x,t)$ be the weak solution to the problem (4.1). As in above sections, we intend to study the blow-up phenomena and quenching behavior of the solution to problem (4.1). We take $\gamma = 1$ for simplicity since similar results can be obtained for any positive γ.

We define auxiliary functions

$$E(u(t)) = \int_\Omega (-\frac{1}{p}|\Delta u|^p - \int_0^u f(|\tau|)d\tau)dx, h(t) = \frac{1}{2}\int_\Omega u^2 + |\nabla u|^2 dx, H(t) = \int_0^t h(s)ds$$

where u is the solution of (4.1). We first start by noticing that each solution of the problem above has the property

$$\int_\Omega u dx = 0$$

because the integral in the right hand side of the first equation in (4.1) is 0 and

$$\begin{aligned}\frac{d}{dt}(\int_\Omega udx) &= \int_\Omega u_t dx \\ &= \int_\Omega \Delta(|\Delta u|^{p-2}\Delta u) + \Delta u_t dx = 0\end{aligned}$$

Hence, by the initial condition $\int_\Omega u_0 dx = 0$, we have $\int_\Omega udx = 0$. Since $W^{2,p}(\Omega) \hookrightarrow L^2(\Omega)$ and $\int_\Omega udx = 0$. The optimal embedding constant B exists such that

$$\| u \|_2 \leq B \| \Delta u \|_p . \tag{4.3}$$

Furthermore, it is not difficult to obtain the following equality

$$\int_0^t \int_\Omega |u_s|^2 + |\nabla u_s|^2 dxds + E(t) = E(0) \tag{4.4}$$

According to this formula, if the initial energy $E(u_0)$ is non-positive, then $E(u(t))$ is non-positive for all $t > 0$. In the case of generalized convex functions of order α, we have

$$C \int_\Omega uf(| u |)dx \geq \int_\Omega \int_0^u f(| t |)dtdx \geq \frac{1}{p} \int_\Omega | \Delta u |^p dx. \tag{4.5}$$

where $C = \frac{1}{1+\alpha}$. The main results of this section are the following theorems.

Theorem 4.1. Let u be the solution of problem (4.1). Assume that $f : [0, \infty) \mapsto [0, \infty)$ is a locally Lipschitz function belonging to the class GC_α, with $p > 1+\alpha$, and let u be a solution of the problem (4.1) corresponding to an initial data $u_0 \in C(\bar{\Omega})$, u_0 not identically zero. If $E(u_0) \leq 0$, then there is $T > 0$ such that

$$\limsup_{t\to T^-} h(t) = \infty$$

The proof of Theorem 4.1 needs a preparation.

Lemma 4.1. Under the assumptions of Theorem 4.1, with $C = \frac{1}{\alpha+1}$, the two auxiliary functions $h(t)$ and $H(t)$ verify the following two conditions:

$$h'(t) \geq \frac{1}{C} \int_0^t \int_\Omega u_t^2 + |\nabla u_t|^2 dxdt; \tag{4.6}$$

$$\frac{1}{2C}(H'(t) - H'(0))^2 \leq H(t)H''(t). \tag{4.7}$$

Proof. By differentiation and the generalized convexity (4.5), we obtain

$$\begin{aligned}h'(t) &= \int_\Omega uu_t + |\nabla u||\nabla u_t|dx \\ &= \int_\Omega u(\Delta(|\Delta u|^{p-2}\Delta u) + f(|u|))dx \\ &= \int_\Omega |\Delta u|^p + uf(|u|)dx \\ &\geq \int_\Omega (|\Delta u|^p + \frac{1}{C}\int_0^u f(t)dt)dx \\ &= -\frac{1}{C}\int_\Omega (-\frac{1}{p}|\Delta u|^p - \int_0^u f(|\tau|)d\tau)dx + (-\frac{1}{Cp} + 1)\int_\Omega |\Delta u|^p dx.\end{aligned} \tag{4.8}$$

Hence,

$$\begin{aligned} h'(t) &\geq -\frac{1}{C}E(u) + (-\frac{1}{Cp}+1)\int_\Omega |\Delta u|^p dx \\ &\geq -\frac{1}{C}E(u) \\ &= -\frac{1}{C}E(u_0) + \frac{1}{C}\int_0^t\int_\Omega u_t^2 + |\nabla u_t|^2 dxdt \\ &\geq \frac{1}{C}\int_0^t\int_\Omega u_t^2 + |\nabla u_t|^2 dxdt. \end{aligned}$$

Since

$$\begin{aligned} H'(t) - H'(0) &= \int_0^t h'(s)ds = \int_0^t\int_\Omega uu_t + |\nabla u||\nabla u_t| dxdt \\ &\leq (\int_0^t\int_\Omega u^2 dxdt)^{1/2}(\int_0^t\int_\Omega u_t^2 dxdt)^{1/2} + (\int_0^t\int_\Omega |\nabla u|^2 dxdt)^{1/2}(\int_0^t\int_\Omega |\nabla u_t|^2 dxdt)^{1/2} \\ &\leq (H(t))^{1/2}(2Ch'(t))^{1/2} = (2CH(t)H''(t))^{1/2}, \end{aligned}$$

by (4.8) we infer that

$$H'(t) - H'(0) = \int_o^t h'(s)ds \geq 0,$$

and thus

$$\frac{1}{2C}(H'(t) - H'(0))^2 \leq H(t)H''(t). \quad \square$$

Proof of Theorem 4.1. Assume the contrary, that the solution $u(x,t)$ exists for all $T > 0$. For any $t_0 > 0$, we claim that

$$\int_0^{t_0}\int_\Omega |u_s|^2 + |\nabla u_s|^2 dxds > 0. \tag{4.9}$$

Otherwise, there exists a $t_0 > 0$ such that $\int_0^{t_0}\int_\Omega |u_s|^2 + |\nabla u_s|^2 dxds = 0$, and hence $u_t = \nabla u_t = 0$ for a.e.$(x,t) \in \Omega \times (0,t_0)$. Then it follows from (4.8) that $\int_\Omega |\Delta u|^p dx = \int_\Omega -uf(|u|)dx$ for a.e. $t \in (0,t_0)$, thus we get from (4.4) that

$$E(t) \geq (-\frac{1}{p} + C)\int_\Omega |\Delta u|^p dx$$

for a.e. $t \in (0,t_0)$, which combines $E(t) \leq E(0) \leq 0$ and $p > 1+\alpha$ imply $\int_\Omega |\Delta u|^p dx = 0$ for a.e. $t \in (0,t_0)$. By (4.3), we have $\|u(\cdot,t)\|_2 = 0$ for a.e. $t \in (0,t_0)$. Furthermore, since $u \in C([0,t_0], L^2(\Omega))$, we obtain $\|u(\cdot,t)\|_2 = 0$ for all $t \in [0,t_0]$, especially $\|u_0\|_2 = 0$, which contradicts to the assumption $\|u_0\| > 0$. Then (4.9) follows.

Fix $t_0 > 0$, and let $\delta = \int_0^{t_0}\int_\Omega |u_s|^2 + |\nabla u_s|^2 dxds$. By (4.9), we know that δ is a positive constant. Integrating (4.6) over (t_0,t), we obtain

$$\begin{aligned} h(t) &\geq h(t_0) + \frac{1}{C}\int_{t_0}^t\int_0^s\int_\Omega |u_\tau|^2 + |\nabla u_\tau|^2 dxd\tau ds \\ &\geq \int_{t_0}^t\int_0^s\int_\Omega |u_\tau|^2 + |\nabla u_\tau|^2 dxd\tau ds \geq \delta(t-t_0). \end{aligned}$$

Hence,

$$\lim_{t\to\infty} H'(t) = \lim_{t\to\infty} h(t) = +\infty.$$

which yields, for each $\beta \in (0, \frac{1}{C})$, the existence of a number $T_0 > 0$ such that for all $t > T_0$,

$$\beta H'(t)^2 \le \frac{1}{C}(H'(t) - H'(0))^2.$$

Now, by (4.7) we obtain

$$\beta H'(t)^2 \le 2H(t)H''(t).$$

We will show, by considering the function $G(t) = H(t)^{-q}$, for a suitable $q > 0$, that the last inequality leads to a contradiction. In fact,

$$\begin{aligned} G''(t) &= qH(t)^{-q-2}((q+1)(H'(t))^2 - H(t)H''(t)) \\ &\le qH(t)^{-q-2}(\frac{2(q+1)}{\beta} - 1)H(t)H''(t) \end{aligned}$$

for all $t > T_0$, so that for $\beta \in (0, 1/C)$ and $q \in (0, 1/(2C) - 1)$ with $2(q+1) < \beta < 1/C$, the corresponding function G(t) is concave.

By (4.7), $\lim_{t\to\infty} H(t) = \infty$, whence $\lim_{t\to\infty} G(t) = 0$. Thus G provides an example of a concave and strictly positive function which tends to 0 at infinity, a fact which is not possible. Consequently, u may blow up at some finite time T. Thus completes the proof. □

Moreover, we derive sufficient conditions to ensure the solution of problem (4.1) does not extinct. The main result is as follows.

Theorem 4.2. Let u be the solution of problem (4.1). Assume that $f : [0, \infty) \mapsto [0, \infty)$ is a locally Lipschitz function belonging to the class GC_α, with $p > 1+\alpha$, and let u be a solution of the problem (1.1) corresponding to an initial data $u_0 \in C(\bar{\Omega})$, u_0 not identically zero. If $E(u_0) \le 0$, then the solution to problem(1.1) does not extinct in finite time.

Proof. By differentiation and the generalized convexity (4.5), we obtain

$$\begin{aligned} h'(t) &= \int_\Omega |\Delta u|^p + \int_\Omega uf(u)dx \\ &= -pE(u) - p\int_\Omega \int_0^u f(|\tau|)d\tau dx + \int_\Omega uf(|u|)dx \\ &\ge -pE(u_0) + (1-pC)\int_\Omega uf(|u|)dx \\ &\ge -pE(u_0) + (1-pC)(\int_\Omega u^2)^{\frac{1}{2}}(\int_\Omega f^2 dx)^{\frac{1}{2}} \\ &\ge -pE(u_0) + Ah^{\frac{1}{2}}(t) \end{aligned} \tag{4.10}$$

where $A = (1-pC)(\int_\Omega f^2 dx)^{\frac{1}{2}}$.

Therefore, by Lemma 3.1 and $E(u_0) < 0$, we have

$$h(t) \ge \min[h(0), (\frac{-pE(u_0)}{A})^2], \quad t > 0.$$

Since $h(0) > 0, a > 0, E(u_0) < 0$, we get $h(t) > 0$ for all $t > 0$. □

5 Conclusion

Throughout this paper, we have studied the blow-up and non-extinction properties of a p-biharmonic parabolic equation with nonlocal nonlinearities and Neumann boundary condition. Compared to the results obtained in [1], we extended some previous results. In addition, we considered a different equation (4.1) and under appropriate assumptions on the relations between coefficients α and p we derive similar results on the blow-up and non-extinction behavior.

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Competing Interests

Authors have declared that no competing interests exist.

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