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Asymptotic Behavior of the System of Second Order Nonlinear Difference Equations

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

This paper deals with dynamics of the solutions to the system of second order nonlinear difference equations

$$x_{n+1} = \frac{x_n}{A + y_n y_{n-1}}, \quad y_{n+1} = \frac{y_n}{B + x_n x_{n-1}}, \quad n = 0, 1, \cdots ,$$

where $A \in (0, \infty), x_{-i} \in (0, \infty), y_{-i} \in (0, \infty), i = 0, 1$. Moreover we use the known results to determine the rate of convergence of the solutions of this system. Finally, we give some numerical examples to justify our results.

Keywords: Difference equation; stability; rate of convergence.

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1 Introduction

Difference equations or discrete dynamical systems are diverse field which impacts almost every branch of pure and applied mathematics. One of the reasons is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in population biology, economic, probability theory, genetics psychology, etc. The theory of difference equations occupies a central position in applicable analysis. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. Rational difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations. It is very interesting to investigate the behavior of solutions to rational difference equations and to discuss the local asymptotic stability of their equilibrium points. Recently there has been published quite a lot of works concerning the behavior of positive solutions of systems of difference equations [1-10]. These results are not only valuable in their own right, but also they can provide insight into their differential counterparts.

Kurbanli [3] studied a three-dimensional system of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1},\ y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1},\ z_{n+1} = \frac{z_{n-1}}{y_n z_{n-1} - 1},$$

where the initial conditions are arbitrary real numbers.

Cinar et al. [4] have obtained the positive solution of the difference equation system

$$x_{n+1} = \frac{m}{y_n},\ y_{n+1} = \frac{p y_n}{x_{n-1} y_{n-1}}.$$

Cinar [5] has obtained the positive solution of the difference equation system

$$x_{n+1} = \frac{1}{y_n},\ y_{n+1} = \frac{y_n}{x_{n-1} y_{n-1}}.$$

Clark and Kulenovic [6] investigated the system of rational difference equations

$$x_{n+1} = \frac{x_n}{a + c y_n}, \quad y_{n+1} = \frac{y_n}{b + d x_n}, \quad n = 0, 1, \cdots,$$

where $a, b, c, d \in (0, \infty)$ and the initial conditions x_0 and y_0 are arbitrary nonnegative numbers.

Zhang, Yang and Liu [7] investigated the global behavior for a system of the following third order nonlinear difference equations.

$$x_{n+1} = \frac{x_{n-2}}{B + y_{n-2} y_{n-1} y_n}, \quad y_{n+1} = \frac{y_{n-2}}{A + x_{n-2} x_{n-1} x_n},$$

where $A, B \in (0, \infty)$, and initial values $x_{-i}, y_{-i} \in (0, \infty), i = 0, 1, 2$.

Ibrahim [9] has obtained the positive solution of the difference equation system in the modeling competitive populations.

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1} y_n + \alpha}, \quad y_{n+1} = \frac{y_{n-1}}{y_{n-1} x_n + \beta}.$$

Din et al. [10] studied the global behavior of positive solution to the fourth-order rational difference equations

$$x_{n+1} = \frac{\alpha x_{n-3}}{\beta + \gamma y_n y_{n-1} y_{n-2} y_{n-3}}, \quad y_{n+1} = \frac{\alpha_1 y_{n-3}}{\beta_1 + \gamma_1 x_n x_{n-1} x_{n-2} x_{n-3}}.$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1$ and the initial conditions $x_{-i}, y_{-i}, i = 0, 1, 2, 3$, are positive real numbers.

Although difference equations are sometimes very simple in their forms, they are extremely difficult to understand thoroughly the behavior of their solutions. In [11], Kocic and Ladas have studied global behavior of nonlinear difference equations of higher order. Similar nonlinear systems of difference equations were investigated (see [12-20]).

Our aim in this paper is to investigate the solutions, stability character and asymptotic behavior of the system of difference equations

$$x_{n+1} = \frac{x_n}{A + y_n y_{n-1}}, \quad y_{n+1} = \frac{y_n}{B + x_n x_{n-1}}, \quad n = 0, 1, \cdots, \tag{1.1}$$

where $A, B \in (0, \infty)$ and initial conditions $x_i, y_i \in (0, \infty), i = -1, 0$.

2 Main Results

Let I_x, I_y be closed intervals of real numbers and $f : I_x^2 \times I_y^2 \to I_x, g : I_x^2 \times I_y^2 \to I_y$ be continuously differentiable functions. Then for every initial conditions $(x_i, y_i) \in I_x \times I_y (i = -1, 0)$, the system of difference equations

$$\begin{cases} x_{n+1} = f(x_n, x_{n-1}, y_n, y_{n-1}), \\ y_{n+1} = g(x_n, x_{n-1}, y_n, y_{n-1}), \end{cases} n = 0, 1, 2, \cdots, \tag{2.1}$$

has a unique solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$. A point $(\bar{x}, \bar{y}) \in I_x \times I_y$ is called an equilibrium point of (2.1) if $\bar{x} = f(\bar{x}, \bar{x}, \bar{y}, \bar{y}), \bar{y} = g(\bar{x}, \bar{x}, \bar{y}, \bar{y})$, i. e., $(x_n, y_n) = (\bar{x}, \bar{y})$ for all $n \geq 0$.

Definition 2.1 *Assume that $(\bar{x}, \bar{y})$ be a fixed point of (2.1). Then*

(i) $(\bar{x}, \bar{y})$ is said to be stable relative to $I_x \times I_y$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any initial conditions $(x_i, y_i) \in I_x \times I_y (i = -1, 0)$, with $\sum_{i=-1}^{0} |x_i - \bar{x}| < \delta, \sum_{i=-1}^{0} |y_i - \bar{y}| < \delta$, implies $|x_n - \bar{x}| < \varepsilon, |y_n - \bar{y}| < \varepsilon$.
(ii) $(\bar{x}, \bar{y})$ is called an attractor relative to $I_x \times I_y$ if for all $(x_i, y_i) \in I_x \times I_y (i = -1, 0)$, $\lim_{n\to\infty} x_n = \bar{x}, \lim_{n\to\infty} y_n = \bar{y}$.
(iii) $(\bar{x}, \bar{y})$ is called asymptotically stable relative to $I_x \times I_y$ if it is stable and an attractor.
(iv) Unstable if it is not stable.

Theorem 2.1.[11] *Assume that $X(n + 1) = F(X(n)), n = 0, 1, \cdots$, is a system of difference equations, and $\overline{X}$ is the equilibrium point of this system i.e., $F(\overline{X}) = \overline{X}$. If all eigenvalues of the Jacobian matrix J_F, evaluated at $\overline{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\overline{X}$ is locally asymptotically stable. If one of them has modulus greater than one then $\overline{X}$ is unstable.*

Clearly, system (1.1) has always trivial equilibrium $(0, 0)$. If $0 < A < 1, 0 < B < 1$, system (1.1) has a unique positive equilibrium point $(\sqrt{1 - B}, \sqrt{1 - A})$.

Theorem 2.2. *Consider system (1.1), then the following statements are true:*

(i) If $0 < A < 1, 0 < B < 1$, then the trivial equilibrium point $(0, 0)$ is unstable.
(ii)If $A > 1, B > 1$, then the trivial equilibrium point $(0, 0)$ is locally asymptotically stable.

Proof. We can obtain easily the linearized system of (1.1) about the positive equilibrium $(0, 0)$ is

$$\Phi_{n+1} = B\Phi_n \tag{2.2}$$

where

$$\Phi_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}, \quad B = \frac{\partial F}{\partial X_n}|_{(0,0)} = \begin{pmatrix} \frac{1}{A} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{B} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic equation of (2.2) is

$$\lambda^2(\lambda - \frac{1}{A})(\lambda - \frac{1}{B}) = 0 \tag{2.3}$$

(i) If $0 < A < 1, 0 < B < 1$, This shows that two roots of characteristic equation lie outside unit disk. So the trivial equilibrium point $(0, 0)$ is a repeler, i.e. it is unstable.
(ii) If $A > 1, B > 1$, The all roots of characteristic equation lie inside unit disk. So the trivial equilibrium point $(0, 0)$ is a source, i.e. it is locally asymptotically stable.

Theorem 2.3 *Suppose that* $A > 1, B > 1$ *hold, then the trivial equilibrium point* $(0, 0)$ *is globally asymptotically stable.*

Proof. For $A > 1, B > 1$, from (ii) of Theorem 2.2, the equilibrium point $(0, 0)$ is locally asymptotically stable. From (1.1), it is easy to see every positive solution (x_n, y_n) is bounded. i.e. $0 < x_n < x_0, 0 < y_n < y_0$. Now it is sufficient to prove that the sequences $\{x_n\}$ and $\{y_n\}$ are decreasing. From (1.1), we have

$$\frac{x_{n+1}}{x_n} = \frac{1}{A + y_n y_{n-1}} \leq \frac{1}{A} < 1, \quad \frac{y_{n+1}}{y_n} = \frac{1}{B + x_n x_{n-1}} \leq \frac{1}{B} < 1.$$

This implies that the sequences $\{x_n\}$ and $\{y_n\}$ are decreasing. Hence $\lim_{n\to\infty} x_n = 0, \lim_{n\to\infty} y_n = 0$. Therefore the trivial equilibrium point $(0, 0)$ is globally asymptotically stable.

Theorem 2.4 *Assume that* $0 < A < 1, 0 < B < 1$. *Then the positive equilibrium point* $(\bar{x}, \bar{y}) = (\sqrt{1 - B}, \sqrt{1 - A})$ *is locally unstable.*

Proof. We can obtain easily the linearized system of (1.1) about the positive equilibrium $(\bar{x}, \bar{y})$ is

$$\Phi_{n+1} = G\Phi_n \tag{2.4}$$

where $\Phi_n = (x_n, x_{n-1}, y_n, y_{n-1})^T$,

$$G = \frac{\partial F}{\partial X_n}|_{(\bar{x},\bar{y})} = \begin{pmatrix} \frac{1}{A} & 0 & -\bar{x}\bar{y} & -\bar{x}\bar{y} \\ 1 & 0 & 0 & 0 \\ -\bar{x}\bar{y} & -\bar{x}\bar{y} & \frac{1}{B} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ denote the 4 eigenvalues of Matrix G. Let $D = \mathrm{diag}(d_1, d_2, d_3, d_4), d_i \neq 0 (i = 1, 2, 3, 4)$ be a diagonal matrix,

Clearly D is invertible. Computing DGD^{-1}, we obtained

$$DGD^{-1} = \begin{pmatrix} \frac{1}{A} & 0 & -\frac{d_1}{d_3}\bar{x}\bar{y} & -\frac{d_1}{d_4}\bar{x}\bar{y} \\ \frac{d_2}{d_1} & 0 & 0 & 0 \\ -\frac{d_3}{d_1}\bar{x}\bar{y} & -\frac{d_3}{d_2}\bar{x}\bar{y} & \frac{1}{B} & 0 \\ 0 & 0 & \frac{d_4}{d_3} & 0 \end{pmatrix}$$

It is well known that G has the same eigenvalues as DGD^{-1}, we obtain that

$$\begin{aligned}\max_{1\leq k\leq 4}|\lambda_k| &= \|DGD^{-1}\| \\ &= \max\left\{d_2d_1^{-1}, d_4d_3^{-1}, \frac{1}{A}+\frac{d_1}{d_3}\bar{x}\bar{y}+\frac{d_1}{d_4}\bar{x}\bar{y},\right. \\ &\left.\frac{1}{B}+\frac{d_3}{d_1}\bar{x}\bar{y}+\frac{d_3}{d_2}\bar{x}\bar{y}\right\} \\ &> 1\end{aligned}$$

It follows from Theorem 2.1 [11] that the positive equilibrium points $(\bar{x},\bar{y})$ is locally unstable.

3 Rate of Convergence

In this section we will determine the rate of convergence of a solution that converges to the equilibrium point $(0,0)$ of the system (1.1). The following result gives the rate of convergence of solution of a system of difference equations

$$X_{n+1}=[P+Q(n)]X_n \tag{3.1}$$

where X_n is a four dimensional vector, $P\in C^{4\times 4}$ is a constant matrix, $Q: Z^{+}\to C^{4\times 4}$ is a matrix function satisfying

$$\|Q(n)\|\to 0, \text{when } n\to\infty. \tag{3.2}$$

where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm.

Theorem 3.1.[21] *Assume that condition (3.2) hold, if X_n is a solution of (3.1), then either $X_n=0$ for all large n or*

$$\rho=\lim_{n\to\infty}\sqrt[n]{\|X_n\|} \tag{3.3}$$

or

$$\rho=\lim_{n\to\infty}\frac{\|X_{n+1}\|}{\|X_n\|} \tag{3.4}$$

exists and is equal to the moduls of one the eigenvalues of the matrix P.

Assume that $\lim_{n\to\infty}x_n=\bar{x}, \lim_{n\to\infty}y_n=\bar{y}$, we will find a system of limiting equations for the system (1.1). The error terms are given as

$$\begin{cases} x_{n+1}-\bar{x}=\sum_{i=0}^{1}P_i(x_{n-i}-\bar{x})+\sum_{i=0}^{1}Q_i(y_{n-i}-\bar{y}) \\ y_{n+1}-\bar{y}=\sum_{i=0}^{1}R_i(x_{n-i}-\bar{x})+\sum_{i=0}^{1}S_i(y_{n-i}-\bar{y})\end{cases}$$

Set $e_n^1=x_n-\bar{x}, e_n^2=y_n-\bar{y}$, therefore it follows that

$$\begin{cases} e_{n+1}^1=\sum_{i=0}^{1}P_ie_{n-i}^1+\sum_{i=0}^{1}Q_ie_{n-i}^2 \\ e_{n+1}^2=\sum_{i=0}^{1}R_ie_{n-i}^1+\sum_{i=0}^{1}S_ie_{n-i}^2\end{cases}$$

where

$$P_0=\frac{1}{A+y_ny_{n-1}}, P_1=0, Q_0=-\frac{x_ny_{n-1}}{(A+y_ny_{n-1})^2}, Q_1=-\frac{x_ny_n}{(A+y_ny_{n-1})^2}.$$
$$R_0=-\frac{x_{n-1}y_n}{(B+x_nx_{n-1})^2}, R_1=-\frac{x_ny_n}{(B+x_nx_{n-1})^2}, S_0=\frac{1}{B+x_nx_{n-1}}, S_1=0.$$

Now it is clear that

$$\lim_{n\to\infty}P_0=\frac{1}{A}, \lim_{n\to\infty}Q_0=\lim_{n\to\infty}Q_1=0, \lim_{n\to\infty}R_0=\lim_{n\to\infty}R_1=0, \lim_{n\to\infty}S_0=\frac{1}{B}.$$

Hence, the limiting system of error terms at $(0,0)$ can be written as

$$E_{n+1} = GE_n \tag{3.5}$$

where $E_n = (e_n^1, e_{n-1}^1, e_n^2, e_{n-1}^2)^T$, and

$$G = J_F(0,0) = (d_{ij})_{4\times 4} = \begin{pmatrix} \frac{1}{A} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{B} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Using Theorem 3.1, we have the following result

Theorem 3.2. *Assume that $A > 1, B > 1$, and $\{(x_n, y_n)\}$ be a positive solution of the system (1.1). Then, the error vector E_n of every solution of (1.1) satisfies both of the following asymptotic relations*

$$\lim_{n\to\infty} \sqrt[n]{\|E_n\|} = |\lambda J_F(0,0)|, \lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|} = |\lambda J_F(0,0)|.$$

where $\lambda J_F(0,0)$ is equal to the moduls of one the eigenvalues of the Jacobian matrix evaluted at the equilibrium $(0,0)$.

4 Numerical Examples

In order to illustrate the results of the previous sections and to support our theoretical discussions, we consider some interesting numerical examples in this section.

Example 4.1. *Consider the system (1.1) with initial conditions $x_{-1} = 0.8, x_0 = 1.2, y_{-1} = 1.8, y_0 = 2.2$, Moreover, choosing the parameters $A = 1.7, B = 1.5$. Then system (1.1) can be written as*

$$x_{n+1} = \frac{x_n}{1.7 + y_n y_{n-1}}, \quad y_{n+1} = \frac{y_n}{1.5 + x_n x_{n-1}} \tag{4.1}$$

The plot of system (4.1) is shown in Fig. 1.

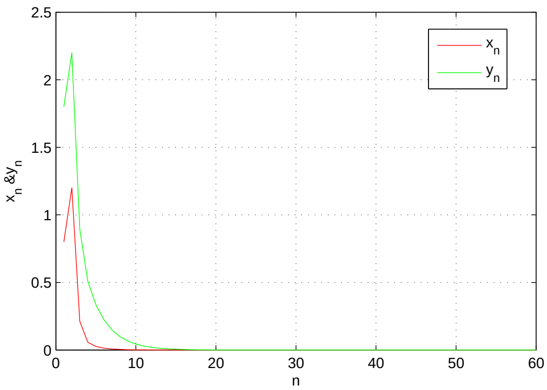

Fig. 1. The plot of system (4.1)

Example 4.2. *Consider the system (1.1) with initial conditions $x_{-1} = 4.8, x_0 = 4.6, y_{-1} = 5.8, y_0 = 5.2$, Moreover, choosing the parameters $A = 0.7, B = 0.8$. Then system (1.1) can be written as*

$$x_{n+1} = \frac{x_n}{0.7 + y_n y_{n-1}}, \quad y_{n+1} = \frac{y_n}{0.8 + x_n x_{n-1}} \tag{4.2}$$

The plot of system (4.2) is shown in Fig. 2.

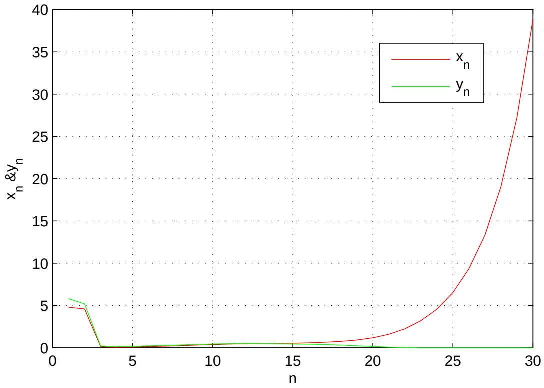

Fig. 2. The plot of system (4.2)

5 Conclusions

In this paper, the dynamical behavior of second-order discrete system is studied. It conclude that: (i) the positive equilibrium point $(\sqrt{1-B}, \sqrt{1-A})$ is locally unstable if $0 < A < 1, 0 < B < 1$. (ii) the trivial equilibrium point $(0, 0)$ is globally asymptotically stable if $A > 1, B > 1$. Some numerical examples are provided to support theoretical results.

Competing Interests

Author has declared that no competing interests exist.

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