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Linear Maps Preserving Rank-additivity and Rank-sum-minimal on Tensor Products of Matrix Spaces

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Authors' contributions

This work was carried out in collaboration among all authors. Author LG designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors YZ and JX managed the analyses of the study. Author YZ managed the literature searches. All authors read and approved the final manuscript.

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Abstract

The problems of characterizing maps that preserve certain invariant on given sets are called the preserving problems, which have become one of the core research areas in matrix theory. If for any $A_1 \otimes \cdots \otimes A_k, B_1 \otimes \cdots \otimes B_k \in M_{n_1} \otimes \cdots \otimes M_{n_k}$, a linear map, $\phi : M_{n_1} \otimes \cdots \otimes M_{n_k} \to M_{n_1} \otimes \cdots \otimes M_{n_k}$, as $R(A_1 \otimes \cdots \otimes A_k + B_1 \otimes \cdots \otimes B_k) = R(A_1 \otimes \cdots \otimes A_k) + R(B_1 \otimes \cdots \otimes B_k)$ established, there is $R(\phi(A_1 \otimes \cdots \otimes A_k + B_1 \otimes \cdots \otimes B_k)) = R(\phi(A_1 \otimes \cdots \otimes A_k)) + R(\phi(B_1 \otimes \cdots \otimes B_k))$ we say that ϕ preserves the rank-additivity. If for any $A_1 \otimes \cdots \otimes A_k, B_1 \otimes \cdots \otimes B_k \in M_{n_1} \otimes \cdots \otimes M_{n_k}$, and a linear map, $\phi : M_{n_1} \otimes \cdots \otimes M_{n_k} \to M_{n_1} \otimes \cdots \otimes M_{n_k}$, as $R(A_1 \otimes \cdots \otimes A_k + B_1 \otimes \cdots \otimes B_k) = \left| R(A_1 \otimes \cdots \otimes A_k) - R(B_1 \otimes \cdots \otimes B_k) \right|$ established, there is $R(\phi(A_1 \otimes \cdots \otimes A_k + B_1 \otimes \cdots \otimes B_k)) = \left| R(\phi(A_1 \otimes \cdots \otimes A_k)) - R(\phi(B_1 \otimes \cdots \otimes B_k)) \right|$ we say that ϕ rank-sum-miminal. In this paper, we characterize the form of linear mapping ϕ.

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1 Introduction

1.1 Notations

Let F be a field, F^* be the set of all non-zero elements in F. Let V be the matrices space over F. For any $A \in V$, $R(A)$ denotes the rank of the matrix A. Let $M_{m\times n}$ be the set of all $m \times n$ matrices. When $m = n$, $M_{m\times n}$ is abbreviated as M_n. E_{ij} denotes the matrix with 1 at the (i, j) entry and 0 elsewhere, the order being determined by context. The Γ denotes the set of all tensor product matrices of rank 1 in $M_{n_1} \otimes \cdots \otimes M_{n_k}$. Let Θ denotes the set of the tensor product matrices satisfying the rank-additivity in $M_{n_1} \otimes \cdots \otimes M_{n_k}$, and Θ_- denotes the set of the tensor product matrices satisfying the rank-sum-minimal in $M_{n_1} \otimes \cdots \otimes M_{n_k}$.

Many scholars have characterized linear maps preserving rank-additivity and rank-sum-minimal on the general matrix space. For example, Alexander Guteran [1] and Beasley [2] respectively described the linear maps that preserving the rank-additivity and the rank-sum-minimal on the n order matrix space and the $m \times n$ matrix space. Zhang Xian [3] further generalized the results of [2]. Then, Zhang Xian [4] discussed the linear map of the rank-additivity and the rank-sum-minimal on the symmetric matrix space. Tang Xiaomin [5] characterized the linear map of the rank-additivity on the Hermitian matrix space. At present, the research on the linear preserving problem of rank-additivity and the rank-sum-minimal on the general matrix space has been basically improved.

Due to personal scientific research interests, I like to study preserving problems. On the one hand, due to its theoretical value, its conclusions are often very refined. On the other hand, it is due to its practical application value. It has a wide range of applications in the fields of differential equations, system control, mathematical statistics, etc. In the study of differential equations, in order to simplify the problem, people sometimes change it before solving a problem, from one system to another. The system makes it more conducive to problematic research.

The tensor product matrix space, as a special matrix, the advantage of the matrix tensor product is that it is an operation defined between two arbitrarily sized matrices. Unlike the general matrix product, which needs to be limited by the number of rows and columns. It has a wide range of applications and the research is more meaningful. Since the tensor product of the matrix can also be used in data encryption technology, the limitation of adding tensor product to the linear retention problem has potential applications for future quantum information science [6]. Therefore, it is particularly important to study the problem of preserving the tensor product matrix space. The research results about preserving the tensor product matrix space are not too many. Zheng BD [7] characterizes the linear maps preserving rank of the tensor products of matrices. Zejun Huang [8] further studied the linear rank preservers of tensor products of rank one. These laid the foundation for subsequent research. This paper not only gives a supplementary to the preserving rank-additivity and rank-sum-miminal for previous studies, but also a preparation for the subsequent study of the preserving rank problems, which will be mentioned in the conclusion, and it enriches the existing theory of preserving problem in matrix tensor product space.

1.2 Preliminaries

If a pair of matrices $A, B \in V$ satisfy $R(A + B) = R(A) + R(B)$ or

$R(A+B)=\left|R(A)-R(B)\right|$, then it is said to be rank-additivity or rank-sum-minimal. For the linear map ϕ on V and a pair of matrices A,B, if $R(A+B)=R(A)+R(B)$ are deduced from $R\phi(A+B)=R\phi(A)+R\phi(B)$, we say that ϕ preserves the rank-additivity. If $R(A+B)=\left|R(A)-R(B)\right|$ are deduced from $R\phi(A+B)=\left|R\phi(A)-R\phi(B)\right|$ we say that ϕ preserves the rank-sum-minimal. For any $A_i \in M_{n_i}(i=1,\cdots,k)$, we call a linear map π on $M_{n_1}\otimes\cdots\otimes M_{n_k}$ canonical, if $\pi(A_1\otimes\cdots\otimes A_k)=\tau_1(A_1)\otimes\cdots\otimes\tau_k(A_k)$, where $\tau_i : M_{n_i}\to M_{n_i}$, is either the identity map or the transposition map. In this case, we note $\pi=\tau_1\otimes\cdots\otimes\tau_k$.

The following theorems and lemmas are required for the two theorems in this paper.

Theorem 1. (7, main theorem) For any $A_1\otimes\cdots\otimes A_k \in M_{n_1}\otimes\cdots\otimes M_{n_k}$,(where $k\geq 1, n_k\geq 2$) and a linear map $\phi : M_{n_1}\otimes\cdots\otimes M_{n_k}\to M_m$, preserves rank of tensor products of matrices, i.e.

$$R\phi(A_1\otimes\cdots\otimes A_k)=R(A_1\otimes\cdots\otimes A_k)$$

if and only if there exist invertible matrices $P,Q\in M_m$ and a canonical map π on $M_{n_1}\otimes\cdots\otimes M_{n_k}$ such that

$$\phi(X)=P\begin{bmatrix}\pi(X) & 0\\ 0 & 0\end{bmatrix}Q$$

for all $X\in M_{n_1}\otimes\cdots\otimes M_{n_k}$.

Lemma 1. (3, Lemma1) $(A,B)\in\Theta$ for any $A,B\in M_{m\times n}$ if and only if there exist invertible matrices

$P\in M_m$ and $Q\in M_n$ such that

$$A=P(I_u\oplus O)Q\ , B=P(O\oplus I_v)Q\ \ , u+v\leq\min(m,n).$$

Lemma 2. $(A_1\otimes\cdots\otimes A_k, B_1\otimes\cdots\otimes B_k)\in\Theta$ for any $A_1\otimes\cdots\otimes A_k$, $B_1\otimes\cdots\otimes B_k\in M_{n_1}\otimes\cdots\otimes M_{n_k}$ if and only if $(A_1\otimes\cdots\otimes A_k, h(B_1\otimes\cdots\otimes B_k))\in\Theta$ for any $h\in F$.

Proof: The result can be deduced immediately from the rank property of the matrix.

2 Main Results

Theorem 2. For any $A_1\otimes\cdots\otimes A_k, B_1\otimes\cdots\otimes B_k\in M_{n_1}\otimes\cdots\otimes M_{n_k}$, a linear map, $\phi : M_{n_1}\otimes\cdots\otimes M_{n_k}\to M_{n_1}\otimes\cdots\otimes M_{n_k}$, as

$$R(A_1 \otimes \cdots \otimes A_k + B_1 \otimes \cdots \otimes B_k) = R(A_1 \otimes \cdots \otimes A_k) + R(B_1 \otimes \cdots \otimes B_k)$$

established, there is

$$R(\phi(A_1 \otimes \cdots \otimes A_k + B_1 \otimes \cdots \otimes B_k)) = R(\phi(A_1 \otimes \cdots \otimes A_k)) + R(\phi(B_1 \otimes \cdots \otimes B_k))$$

then $\phi = 0$, or there exist invertible matrices $P, Q \in M_{n_1} \otimes \cdots \otimes M_{n_k}$ and a canonical map π on $M_{n_1} \otimes \cdots \otimes M_{n_k}$ such that

$$\phi(X) = P\pi(X)Q$$

for all $X \in M_{n_1} \otimes \cdots \otimes M_{n_k}$.

Proof. We distinguish two cases:

(i) Suppose ϕ is not injective. For some $A_1 \otimes \cdots \otimes A_k \in M_{n_1} \otimes \cdots \otimes M_{n_k}$ with $R(A_1 \otimes \cdots \otimes A_k) = s \geq 1$. Without loss of generality, we may assume that $\phi(I_s \oplus O) = 0$. Since $(E_{11} \otimes \cdots \otimes E_{11}, 0 \oplus I_{s-1} \oplus O) \in \Theta$, we have

$$(\phi(E_{11} \otimes \cdots \otimes E_{11}), \phi(0 \oplus I_{s-1} \oplus O)) \in \Theta$$

or equivalently

$R((\phi(E_{11} \otimes \cdots \otimes E_{11}) + \phi(0 \oplus I_{s-1} \oplus O)) = R(\phi(E_{11} \otimes \cdots \otimes E_{11})) + R(\phi(0 \oplus I_{s-1} \oplus O))$ but $\phi(E_{11} \otimes \cdots \otimes E_{11}) + \phi(0 \oplus I_{s-1} \oplus O) = \phi(I_s \oplus O)$, i.e.,

$$R(\phi(I_s \oplus O)) = R(\phi(E_{11} \otimes \cdots \otimes E_{11})) + R(\phi(0 \oplus I_{s-1} \oplus O))$$

Then, according to $\phi(I_s \oplus O) = 0$, we obtain

$$\phi(E_{11} \otimes \cdots \otimes E_{11}) = 0 \tag{1}$$

For any $j_i = 1, \cdots, n_i$ (where $i = 1, \cdots, k$, and $j_1, \cdots, j_k$ can't be 1 at the same time), as

$$(E_{11} \otimes \cdots \otimes E_{11} - E_{1j_1} \otimes \cdots \otimes E_{1j_k}, E_{j_1j_1} \otimes \cdots \otimes E_{j_kj_k} + E_{1j_1} \otimes \cdots \otimes E_{1j_k}) \in \Theta,$$
$$(E_{11} \otimes \cdots \otimes E_{11} + E_{1j_1} \otimes \cdots \otimes E_{1j_k}, E_{j_1j_1} \otimes \cdots \otimes E_{j_kj_k}) \in \Theta$$

we have

$$(\phi(E_{11} \otimes \cdots \otimes E_{11} - E_{1j_1} \otimes \cdots \otimes E_{1j_k}), \phi(E_{j_1j_1} \oplus \cdots \otimes E_{j_kj_k} + E_{1j_1} \otimes \cdots \otimes E_{1j_k})) \in \Theta,$$

$$(\phi(E_{11} \otimes \cdots \otimes E_{11} + E_{1j_1} \otimes \cdots \otimes E_{1j_k}), \phi(E_{j_1j_1} \otimes \cdots \otimes E_{j_kj_k})) \in \Theta$$

or equivalently.

$$\begin{aligned}&R(\phi(E_{11} \otimes \cdots \otimes E_{11} + E_{j_1j_1} \otimes \cdots \otimes E_{j_kj_k}))\\&= R(\phi(E_{11} \otimes \cdots \otimes E_{11} - E_{1j_1} \otimes \cdots \otimes E_{1j_k})) + R(\phi(E_{j_1j_1} \otimes \cdots \otimes E_{j_kj_k} + E_{1j_1} \otimes \cdots \otimes E_{1j_k}))\end{aligned}$$

$$\begin{aligned}&R(\phi(E_{11} \otimes \cdots \otimes E_{11} + E_{j_1j_1} \otimes \cdots \otimes E_{j_kj_k} + E_{1j_1} \otimes \cdots \otimes E_{1j_k}))\\&= R(\phi(E_{11} \otimes \cdots \otimes E_{11} + E_{1j_1} \otimes \cdots \otimes E_{1j_k})) + R(\phi(E_{j_1j_1} \otimes \cdots \otimes E_{j_kj_k}))\end{aligned}$$

By calculation, together with (1), the above two formulas can get

$$R(\phi(E_{j_1j_1} \otimes \cdots \otimes E_{j_kj_k}) = R(\phi(E_{1j_1} \otimes \cdots \otimes E_{1j_k}) + R(\phi(E_{j_1j_1} \otimes \cdots \otimes E_{j_kj_k} + E_{1j_1} \otimes \cdots \otimes E_{1j_k}))$$

$$R(\phi(E_{1j_1} \otimes \cdots \otimes E_{1j_k} + E_{j_1j_1} \otimes \cdots \otimes E_{j_kj_k})) = R(\phi(E_{1j_1} \otimes \cdots \otimes E_{1j_k})) + R(\phi(E_{j_1j_1} \otimes \cdots \otimes E_{j_kj_k}))$$

Then, we have $R(\phi(E_{1j_1} \otimes \cdots \otimes E_{1j_k})) = 0$, i.e. ,

$$\phi(E_{1j_1} \otimes \cdots \otimes E_{1j_k}) = 0 \tag{2}$$

Similarly, for any $p_i = 1, \cdots, n_i$ (where $i = 1, \cdots, k$, and $p_1, \cdots, p_k$ can't be 1 at the same time), then we can get

$$\phi(E_{p_11} \otimes \cdots \otimes E_{p_k1}) = 0 \tag{3}$$

For any $j_i = 1, \cdots, n_i; p_i = 1, \cdots n_i (i = 1, \cdots, k)$, as

$(E_{1j_1} \otimes \cdots \otimes E_{1j_k} - E_{p_1j_1} \otimes \cdots \otimes E_{p_kj_k}, E_{p_11} \otimes \cdots \otimes E_{p_k1} + E_{p_1j_1} \otimes \cdots \otimes E_{p_kj_k}) \in \Theta$ we can get

$$(\phi(E_{1j_1} \otimes \cdots \otimes E_{1j_k} - E_{p_1j_1} \otimes \cdots \otimes E_{p_kj_k}), \ \phi(E_{p_11} \otimes \cdots \otimes E_{p_k1} + E_{p_1j_1} \otimes \cdots \otimes E_{p_kj_k})) \in \Theta$$

or equivalently.

$$\begin{aligned}R(\phi(E_{1j_1} \otimes \cdots \otimes E_{1j_k} + E_{p_11} \otimes \cdots \otimes E_{p_k1}) &= R(\phi(E_{1j_1} \otimes \cdots \otimes E_{1j_k} - E_{p_1j_1} \otimes \cdots \otimes E_{p_kj_k}))\\&+ R(\phi(E_{p_11} \otimes \cdots \otimes E_{p_k1} + E_{p_1j_1} \otimes \cdots \otimes E_{p_kj_k}))\end{aligned}$$

Using (2) and (3), we can get

$$R(\phi(E_{p_1j_1} \otimes \cdots \otimes E_{p_kj_k})) = 0 \tag{4}$$

Therefore, together with(2)-(4) and the linearity of ϕ, we can obtain $\phi = 0$.

(ii) Suppose ϕ is injective.

For the convenience of discussion, let $n = n_1 \cdots n_k$. For any $A_{11} \otimes \cdots \otimes A_{1k} \in \Gamma$, It is easy to see that there are $A_{21} \otimes \cdots \otimes A_{2k}, \cdots, A_{n1} \otimes \cdots \otimes A_{nk} \in \Gamma$ such that

$$(A_{11} \otimes \cdots \otimes A_{1k} + \sum_{t=2}^{n-1} A_{t1} \otimes \cdots \otimes A_{tk},\ \ A_{n1} \otimes \cdots \otimes A_{nk}) \in \Theta$$

As ϕ preserving rank-additivity, we have

$$\begin{aligned}
&R(\phi(A_{11} \otimes \cdots \otimes A_{1k} + \sum_{t=2}^{n} A_{t1} \otimes \cdots \otimes A_{nk}) = \\
&R(\phi(A_{11} \otimes \cdots \otimes A_{1k} + \sum_{t=2}^{n-1} A_{t1} \otimes \cdots \otimes A_{(n-1)\ k})) + R(\phi(A_{n1} \otimes \cdots \otimes A_{nk})) \\
&= R(\phi(A_{11} \otimes \cdots \otimes A_{1k} + \sum_{t=2}^{n-2} A_{t1} \otimes \cdots \otimes A_{(n-2)k})) + R(\phi(A_{(n-1)1} \otimes \cdots \otimes A_{(n-1)k})) + \\
&\quad R(\phi(A_{n1} \otimes \cdots \otimes A_{nk})) \\
&= \cdots \\
&= R(\phi(A_{11} \otimes \cdots \otimes A_{1k})) + \sum_{t=2}^{n} R(\phi(A_{t1} \otimes \cdots \otimes A_{tk})).
\end{aligned}$$

Since $A_{t1} \otimes \cdots \otimes A_{tk} \in \Gamma(t = 1,2,\cdots,n)$, and ϕ is injective, We obtain $R(\phi(A_{11} \otimes \cdots \otimes A_{1k})) = 1$. For $B_1 \otimes \cdots \otimes B_k \in M_{n_1} \otimes \cdots \otimes M_{n_k}$ with arbitrary rank $r(r \geq 2)$, it is

obvious that there exist matrices

$$B_{11} \otimes \cdots \otimes B_{1k}, \cdots, B_{r1} \otimes \cdots \otimes B_{rk} \in \Gamma$$

such that

$$B_1 \otimes \cdots \otimes B_k = \sum_{t=1}^{r} B_{t1} \otimes \cdots \otimes B_{tk}$$

Since $(\sum_{t=2}^{r-1} B_{t1} \otimes \cdots \otimes B_{tk}, B_{r1} \otimes \cdots \otimes B_{rk}) \in \Theta$, we have

$$\begin{aligned}
R(\phi(B_1 \otimes \cdots \otimes B_k)) &= R(\phi(\sum_{t=1}^{r} B_{t1} \otimes \cdots \otimes B_{tk})) \\
&= R(\phi(\sum_{t=1}^{r-1} B_{t1} \otimes \cdots \otimes B_{tk})) + R(\phi(B_{r1} \otimes \cdots \otimes B_{rk})) \\
&= \ldots \\
&= \sum_{t=1}^{r} R(\phi(B_{t1} \otimes \cdots \otimes B_{tk}))
\end{aligned}$$

Thus, according to $\phi(B_{t1} \otimes \cdots \otimes B_{tk}) \in \Gamma(t = 1,\cdots,r)$, we obtain $R(\phi(B_1 \otimes \cdots \otimes B_k)) = r$.

Combining the above two aspects yields Theorem 1. ϕ preserves the rank of the tensor product matrix, we can know there exist invertible matrices $P, Q \in M_{n_1} \otimes \cdots \otimes M_{n_k}$ and a canonical map π on $M_{n_1} \otimes \cdots \otimes M_{n_k}$ such that

$$\phi(X) = P\pi(X)Q$$

for all $X \in M_{n_1} \otimes \cdots \otimes M_{n_k}$.

The proof is completed.

Theorem 3. For any $A_1 \otimes \cdots \otimes A_k, B_1 \otimes \cdots \otimes B_k \in M_{n_1} \otimes \cdots \otimes M_{n_k}$, and a linear map, $\phi : M_{n_1} \otimes \cdots \otimes M_{n_k} \to M_{n_1} \otimes \cdots \otimes M_{n_k}$, as

$$R(A_1 \otimes \cdots \otimes A_k + B_1 \otimes \cdots \otimes B_k) = \left| R(A_1 \otimes \cdots \otimes A_k) - R(B_1 \otimes \cdots \otimes B_k) \right|$$

established, there is

$$R(\phi(A_1 \otimes \cdots \otimes A_k + B_1 \otimes \cdots \otimes B_k)) = \left| R(\phi(A_1 \otimes \cdots \otimes A_k)) - R(\phi(B_1 \otimes \cdots \otimes B_k)) \right|$$

Then $\phi = 0$, or there exist invertible matrices $P, Q \in M_{n_1} \otimes \cdots \otimes M_{n_k}$ and a canonical map π on $M_{n_1} \otimes \cdots \otimes M_{n_k}$ such that

$$\phi(X) = P\pi(X)Q$$

for all $X \in M_{n_1} \otimes \cdots \otimes M_{n_k}$.

Proof. For any $A_1 \otimes \cdots \otimes A_k, B_1 \otimes \cdots \otimes B_k \in M_{n_1} \otimes \cdots \otimes M_{n_k}$, if

$(A_1 \otimes \cdots \otimes A_k, B_1 \otimes \cdots \otimes B_k) \in \Theta$,

Then, using Lemma 2, for any $h \in F$ we have $(A_1 \otimes \cdots \otimes A_k, h(B_1 \otimes \cdots \otimes B_k)) \in \Theta$.This implies

$(A_1 \otimes \cdots \otimes A_k + h(B_1 \otimes \cdots \otimes B_k), -h(B_1 \otimes \cdots \otimes B_k)) \in \Theta_-$

$(A_1 \otimes \cdots \otimes A_k + h(B_1 \otimes \cdots \otimes B_k), -(A_1 \otimes \cdots \otimes A_k)) \in \Theta_-$ for any $h \in F$.

As ϕ preserves the rank-sum-minimal of the tensor product matrix, we get

$(\phi(A_1 \otimes \cdots \otimes A_k + h(B_1 \otimes \cdots \otimes B_k)), \phi(-h(B_1 \otimes \cdots \otimes B_k)) \in \Theta_-$,

$(\phi(A_1 \otimes \cdots \otimes A_k + h(B_1 \otimes \cdots \otimes B_k)), \phi(-(A_1 \otimes \cdots \otimes A_k))) \in \Theta_-$.

Further calculations, we can get

$$R(\phi(A_1 \otimes \cdots \otimes A_k))$$
$$= \left| R(\phi(A_1 \otimes \cdots \otimes A_k + h(B_1 \otimes \cdots \otimes B_k)) - R(\phi(-h(B_1 \otimes \cdots \otimes B_k))) \right|$$
$$= \left| R(\phi(A_1 \otimes \cdots \otimes A_k) + h\phi(B_1 \otimes \cdots \otimes B_k)) - R(\phi(h(B_1 \otimes \cdots \otimes B_k))) \right| \quad \forall k \in F \quad (5)$$

$$R(\phi(h(B_1 \otimes \cdots \otimes B_k))$$
$$= \left| R(\phi(A_1 \otimes \cdots \otimes A_k + h(B_1 \otimes \cdots \otimes B_k)) - R(\phi(-(A_1 \otimes \cdots \otimes A_k))) \right|$$
$$= \left| R(\phi(A_1 \otimes \cdots \otimes A_k + h\phi(B_1 \otimes \cdots \otimes B_k)) - R(\phi(A_1 \otimes \cdots \otimes A_k)) \right| \quad \forall k \in F \quad (6)$$

Next, we will discuss it in three situations.

(i): Suppose $R(\phi(A_1 \otimes \cdots \otimes A_k) + h_0\phi(B_1 \otimes \cdots \otimes B_k)) \geq R(\phi(h_0(B_1 \otimes \cdots \otimes B_k)))$ for some $h_0 \in F^*$, we have

$R(\phi(A_1 \otimes \cdots \otimes A_k)) = R(\phi(A_1 \otimes \cdots \otimes A_k) + h_0\phi(B_1 \otimes \cdots \otimes B_k)) - R(\phi(h_0(B_1 \otimes \cdots \otimes B_k))$ and hence

$R(\phi(A_1 \otimes \cdots \otimes A_k) + h_0\phi(B_1 \otimes \cdots \otimes B_k)) = R(\phi(A_1 \otimes \cdots \otimes A_k)) + R(\phi(h_0(B_1 \otimes \cdots \otimes B_k))$, i,e.,

$(\phi(A_1 \otimes \cdots \otimes A_k), h_0\phi(B_1 \otimes \cdots \otimes B_k)) \in \Theta$, this, together with Lemma 2, gives

$(\phi(A_1 \otimes \cdots \otimes A_k), \phi(B_1 \otimes \cdots \otimes B_k)) \in \Theta$.

(ii): Suppose $R((\phi(A_1 \otimes \cdots \otimes A_k + h_0\phi(B_1 \otimes \cdots \otimes B_k)) \geq R(\phi(A_1 \otimes \cdots \otimes A_k))$

for some $h_0 \in F^*$ .Then, by a similar argument to Case (i), we have from (6) that $(\phi(A_1 \otimes \cdots \otimes A_k), \phi(B_1 \otimes \cdots \otimes B_k)) \in \Theta$ holds.

(iii): Suppose $R(\phi(A_1 \otimes \cdots \otimes A_k) + h\phi(B_1 \otimes \cdots \otimes B_k)) \leq R(\phi(h(B_1 \otimes \cdots \otimes B_k)))$ and

$R((\phi(A_1 \otimes \cdots \otimes A_k + h\phi(B_1 \otimes \cdots \otimes B_k)) \leq R(\phi(A_1 \otimes \cdots \otimes A_k))$ for any $h \in F^*$. Then by (5) and (6), we have

$$R(\phi(A_1 \otimes \cdots \otimes A_k)) = R(\phi(h(B_1 \otimes \cdots \otimes B_k))) - R(\phi(A_1 \otimes \cdots \otimes A_k) + h\phi(B_1 \otimes \cdots \otimes B_k))$$
$$R(\phi(h(B_1 \otimes \cdots \otimes B_k))) = R(\phi(A_1 \otimes \cdots \otimes A_k)) - R(\phi(A_1 \otimes \cdots \otimes A_k) + h\phi(B_1 \otimes \cdots \otimes B_k))$$

consequently, $R(\phi(A_1 \otimes \cdots \otimes A_k + h\phi(B_1 \otimes \cdots \otimes B_k)) = 0$, i.e.,

$$\phi(A_1 \otimes \cdots \otimes A_k + h\phi(B_1 \otimes \cdots \otimes B_k) = 0.$$

By the arbitrariness of h, we can know $\phi(A_1 \otimes \cdots \otimes A_k) = 0$ and $(\phi(A_1 \otimes \cdots \otimes A_k), \phi(B_1 \otimes \cdots \otimes B_k)) \in \Theta$.

By combining the above three situations, we can obtain that ϕ preserves rank-additivity on tensor products of matrix spaces. Thus, by Theorem 2, we can get

$\phi = 0$, or there exist invertible matrices $P, Q \in M_{n_1} \otimes \cdots \otimes M_{n_k}$ and a canonical map π on $M_{n_1} \otimes \cdots \otimes M_{n_k}$ such that

$$\phi(X) = P\pi(X)Q$$

for all $X \in M_{n_1} \otimes \cdots \otimes M_{n_k}$.

The proof is completed.

3 Conclusions

This paper correctly demonstrates linear maps preserving rank-additivity and rank-sum-minimal on tensor products of matrix spaces. This result has a certain significance for future research on tensor products of matrix spaces. In fact, starting from preserving problem, more problems can be studied. For example, removing the linear condition and change it to the addition condition and the square matrix is changed into a general matrix.

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Competing Interests

Authors have declared that no competing interests exist.

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