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Analytical Solution of the Complex Polymer Equation Systems via the Homogeneous Balance Method

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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ABSTRACT

In this article, we solve analytically the nonlinear Doubly Dispersive Equation (DDE) in (1+1)-D by the homogeneous balance method, introduced to investigate the strain waves propagating in a cylindrical rod in complex polymer systems. The linear dispersion relation plays important role in connecting the frequency of the emitted nonlinear waves with the wave number of the ablating laser beam affecting the polymers with their characteristic parameters. In accordance with the normal dispersion conditions, the resulting solitary wave solutions show the compression characters in the nonlinearly elastic materials namely Polystyrene (PS) and PolyMethylMethAcrylate (PMMA). The ratio between the estimated potential and kinetic energies shows good agreement with the physical situation, and as well in making comparisons with the bell-shaped model conducted in the literature.

Keywords: Polymers; laser ablation; Doubly Dispersive Equation (DDE); travelling wave variable; normal dispersion; strain; homogeneous balance method; soliton solutions.

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1. INTRODUCTION

The Korteweg de-Vries (KdV) is one of the first long wave models which we use in mathematics and physics, many authors, such as Nariboli and Sedove [1] uses the equation of (KdV) to study the far-field structure of waves in cylinders and plates, and tried to solve it for the propagation of longitudinal waves in non-linear elastic medium. Later on the construction of it a study had been conducted in [2], which asked about the possible existence of solitary waves in infinite, homogeneous, isotropic, elastic media. In [3,4] the authors succeeded by pioneering experiments to generate bulk solitary waves in optically transparent polymeric material such as polystyrene (PS) and Plexiglas (PMMA), and showed that splitting of a waveguide leads to fission of bulk solitons in solids. They investigated the strain solitons and showed how to construct them in solids [5], and theoretically proved by using the so called doubly dispersive equation (DDE), longitudinal bulk solitary waves in an elastic rod which is not similar to Boussinesq, but could be said that it is an of this type with two kinds of dispersive terms.

In [5] the propagation of a weakly nonlinear wave in a thin circular rod of radius R proved that the wave equation is not Boussinesq, since the arrangements of real physical experiments including generation, detection and observation of strain solitary waves in solids are investigated.

The studies on strain solitary waves in different polymers shapes via various conditions and materials were conducted in [6,7], where the experimental and theoretical investigations demonstrate the physical possibility of generation and further propagation of bulk strain solitary waves in waveguides made of nonlinearly elastic materials, like glassy polymers, Polystyrene (PS) and Polymethyl Methacrylate (PMMA). In the experimental trials set-up, physics authors in [2,3,5,7,8,9] and recently in [10] had been using and reporting about the laser ablation to remove or destruct polymer materials by vaporization and erosive processes. This process is mainly influenced by the nature of the polymers material and their tendency to absorb energy, such that the wavelength of the ablation laser should have a threshold absorption depth [11].

2. STATEMENT OF THE PHYSICAL PROBLEM AND BASIC EQUATIONS

It is considered that a finite homogenous isotropic cylindrical rod of a nonlinearly elastic compressible material in a system of cylindrical Lagrangian coordinates (x, r, φ) with the x axis coinciding with the rod axis, and $\varphi \in [0, 2\pi]$, $0 \le r \le R$, and $-\frac{l}{2} \le x \le \frac{l}{2}$, where R and x are the rod radius and length respectively, with no torsion, see [5,6]. In particular, the generation and recording of the bulk compression waves performed using Q-switched pulsed ruby lasers (20 ns, 0.5 J), in order to propagate the strain waves in polymeric material, as was described above. The longitudinal wave equation of a finite deformation elastic circular rod describing DDE is [8]

$$v_{tt} - c_o^2 v_{xx} = \frac{1}{2}[\frac{\beta}{\rho} v^2 + \upsilon^2 R^2 (v_{tt} - c_1^2 v_{xx})]_{xx} \ , \qquad (1)$$

where $v(x,t)$ is the strain function, x and t are the space coordinate and time, respectively. This leads to a KdV-type equation under the one-wave restriction, where the nonlinearity coefficient β depends on Murnaghan's modulo (l, m, n) Young's modulus $E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$, Poisson's ratio $\upsilon = \frac{\lambda}{2(\lambda+\mu)}$, and is given by

$$\beta = 3E + 2l(1-2\upsilon)^3 + (1+\upsilon)^2(1-2\upsilon) + 6n\upsilon^3 \ , \qquad (2)$$

where ρ is the density of material; here we have introduced the linear longitudinal wave velocity c_o as the speed of linear P-wave and c_1 as the speed of linear S-waves which one defines as

$$c_o = \sqrt{\frac{E}{\rho}} \quad , \quad c_1 = \sqrt{\frac{\mu}{\rho}} = \frac{c_o}{\sqrt{2(1+\upsilon)}} \quad . \qquad (3)$$

3. METHOD OF SOLUTION

3.1 Dispersion Analysis: Group and Phase Velocities

Several methods were used to solve nonlinear equations, after replacing the independent variables *(x, t)* by a single variable called as the travelling wave variable such that,

$$\zeta = kx - \omega t \ , \qquad (4)$$

where k is the wave number in (rad/m) and ω is the frequency (rad/sec.).

To find the relation between the wave number and the frequency we substitute eq. (4) into eq. (1), after neglecting the nonlinear term in it. In general, the process of neglecting the nonlinear terms is an approximation step in order to follow separately the effect of linearity, so we get the form,

$$v_{tt} - c_o^2 v_{xx} = \frac{1}{2}[\upsilon^2 R^2 (v_{tt} - c_1^2 v_{xx})]_{xx} . \quad (5)$$

Expressing the strain linearly as

$$v(x,t) = v_0 \exp i\zeta . \quad (6)$$

Now we substitute expression (6) in (5) we get the linear dispersion relation

$$c_0{}^2 k^2 - \omega^2 + \frac{1}{2} k^2 R^2 v^2 (c_1{}^2 k^2 - \omega^2) = 0$$

with two branches (±):

$$\omega_{\pm} = \pm \frac{k\sqrt{2c_0^2 + c_1^2 k^2 R^2 \upsilon^2}}{\sqrt{2 + k^2 R^2 \upsilon^2}} \quad (7)$$

Regarding equation (7), as long as the speed and direction of energy transport is determined by the group velocity [12], the difference between the group velocity (*Vgr=* $\partial\omega/\partial k$ *)*

$$v_{gr} = \pm \frac{4c_0^2 + c_1^2 k^2 R^2 v^2 (4 + k^2 R^2 v^2)}{(2 + k^2 R^2 v^2)^{\frac{3}{2}} \sqrt{2c_0^2 + c_1^2 k^2 R^2 v^2}} , \quad (8)$$

and the phase velocity of the individual waves travelling along the rod (*Vph=* ω/k)

$$v_{ph} = \pm \frac{\sqrt{2c_0^2 + c_1^2 k^2 R^2 v^2}}{\sqrt{2 + k^2 R^2 v^2}} , \quad (9)$$

should indicate whether the waves in the system are normally or anomalously dispersive with respect to the chosen laser wave numbers that are affecting the polymer materials under study. As we shall see latter that ω_{-} will satisfy the condition of normal dispersion.

3.2 The Homogeneous Balance Method

The homogeneous balance method [13] is a powerful tool to find solitary wave solutions of nonlinear partial differential equations. Fan and Zhang [14] introduced the homogeneous balance method throughout the search for Bäcklund transformations and obtained more solutions.

Resorting first to (4), we substitute equation (6) in equation (1) and by using the following differentials $\frac{\partial}{\partial t} = -\omega \frac{d}{d\zeta}, \ \frac{\partial}{\partial x} = k\frac{d}{d\zeta}$, we get:

$$(\omega^2 - c_0^2 k^2)\, v_{\zeta\zeta} - \frac{\beta}{2\rho} k^2 (v^2)_{\zeta\zeta} + \upsilon^2 R^2 (\omega^2 k^2 - c_1^2 k^4)\, v_{\zeta\zeta\zeta\zeta} = 0 . \quad (10)$$

It is well known that soliton waves with infinite support are generated as a result of the balance, e.g. in [15,14], between the nonlinear convection term $(v^2)_{\zeta\zeta}$ which causes the steepening of the wave form and the linear dispersion term $v_{\zeta\zeta\zeta\zeta}$ which makes the wave form spread, as the integrable nonlinear KdV Equation in equation (1), this can also be seen in evolution equations such as the Boussinesq and the Schrödinger equations [16,17].

Following the basic rules in [13], we will verify in (10) the homogeneous balance between the highest order derivative terms with nonlinear terms to this system : the highest order derivative term namely $v_{\zeta\zeta\zeta\zeta}$ gives *f+4*, while the highest non-linear term is $(v^2)_{\zeta\zeta} = 2(v\ v_{\zeta\zeta} + (v_\zeta)^2)$, which gives in both terms *2f+2*; hence equating to get *f=2*.

The Complex-Tanh function method is one of those methods, which are widely applied in solving such a problem. Let us expand the strain variable in the truncated series to the power *f* as

$$v(\zeta) = \sum_{f=0}^{f=2} a_f \tanh^f(i\zeta) . \quad (11)$$

This gives

$$v(\zeta) = a_0 + a_1 \tanh(i\zeta) + a_2 \tanh^2(i\zeta) . \quad (12)$$

Using symbolic software Mathematica [18], substituting from (12) into (10) we get a system of algebraic equations corresponding to different powers of the Tanh-function, the case which satisfies the physical situation for a_0, a_1 and a_2 is the following:

$$a_0 = \frac{54k^2\beta - 13c_0^2k^2\rho + 13\rho\omega^2}{13k^2\beta}, \quad a_1 = \frac{\sqrt{\frac{10}{39}}\sqrt{-a_0k^2\beta - c_0^2k^2\rho + \rho\omega^2}}{k\sqrt{\beta}}, \quad a_2 = \frac{a_0k^2\beta + c_0^2k^2\rho - \rho\omega^2}{9k^2\beta}. \quad (13)$$

Accordingly, substituting these coefficients in equ. (12), this will describe the strain in the studied materials by knowing their densities, Murnaghan's numbers, Young's modulus and Poisson ratios.

The authors in [15] among others, studied the physical possibility of the generation and propagation of bulk longitudinal strain solitons in waveguides made of nonlinear elastic polymeric materials; such as polystyrene (PS) and polymethyl methacrylate (PMMA). The strain soliton represents a powerful localized wave capable of transferring elastic energy over considerable distances almost without losses.

4. KINETIC AND POTENTIAL ENERGIES

In terms of the strain function, the kinetic and potential energies which are mentioned in [4,8] are respectively:

$$T = \frac{1}{2}\rho s v_t^2 + \frac{1}{4}\rho\upsilon^2 v_{tt}^2, \quad (14)$$

$$E_p = \frac{s}{2}\left[Ev_x^2 + \frac{\beta}{3}v_x^3 + \frac{\mu s\upsilon^2}{2\pi}v_{xx}^2\right] \quad (15)$$

According to the study [9] and after a long time in [10], they extensively investigated the damage effects of various energetic laser beams on polymers such as PS and PMMA, is known as laser ablation. Their results show that the damage is determined by two types of processes: physical interaction of radiation with the substance and mechanical development of the damage cracks. Therefore, due to laser ablation this enables us to make the signs of the ratio between E_p and T as a measure of micro-structural instabilities in the studied materials according to the wave numbers (wavelength).

Dreiden, Samsonov and Semenova [15] solved the DDE equation independently from the discrete elastic lattice problem as a continuum limit for guided waves in the lattice. Its general solution in terms of the Weierstrass elliptic functions, with proper limit to solitary wave solutions, which describes the relation between strain v as a function of the length x and time t in equation (1).

The solution of the strain is bell-shaped:

$$v = A\cosh^{-2}\left[\frac{1}{L(A)}(x \pm V(A)\,t)\right], \quad (16)$$

where, $V(A)^2 = c^2 + \frac{A\beta}{3\rho}$, $L^2 = \frac{8R^2\nu^2}{3}\left[1 + \frac{3E}{A\beta}\cdot\frac{1+2\nu}{2(1-\nu)}\right]$,

$$c = \sqrt{\frac{E}{\rho}}, \quad c_1 = \frac{c}{\sqrt{2(1+\nu)}}, \quad A = \frac{\nu\beta}{\gamma c(1+c)}, \quad \gamma = \frac{5}{8} \quad (17)$$

Accordingly, they firstly proved experimentally that there is a deformation done in polymeric materials, and the amplitude *A* is proportional to soliton velocity *V* is constant. Secondly, they proved that there always will be at least one discrete eigenvalue, i.e., at least one solitary wave in the transmitted wave field, accompanied by the radiation corresponding to the continuous spectrum.

5. THE RESULTS AND DISCUSSION

In this paper, we have studied the propagation of a nonlinear longitudinal bulk strain wave in a rod of a polymeric materials and comparing the results with the results governed in [15], we get almost the same results with respect to the strain function. It ensures the success of the Tanh-function method applied as a truncated series in the homogeneous balanced equation to find the strain function in polymeric materials under study.

Table 1 summarizes the mechanical characteristics of (PS) and (PMMA), which are important with respect to the generation and propagation of strain solitons in waveguides made of these polymers. Notice that the figures are presented at the best view angles.

While we plot the strain for Ps and PMMA use is made of the data in Table (1), at $0 < t < 10^{-3}$ sec., and -0.025 m $< x < 0.025$ m (the length of the rod=0.05m), introduced in [13].

Table 1. Elastic properties of polymeric materials [6]

	Density	Young modulus	Poisson ratio	Elastic moduli, 3rd order (Murnaghan)	(N/m^2 10^{10})		Sound velocity in a rod
	ρ (Kg/m^2)	E (N/m^{2}10^{10})	υ	l	m	n	C_0 (m/s)
PS	1060	0.37	0.34	-1.89	-1.33	-1	1870
PMMA	1160	0.5	0.34	-1.09	-0.77	-0.14	2080

The wave number k plays important role in connecting the frequency of the emitted nonlinear waves with the wave number of the laser beam ablating the polymers together with their characteristic parameters.

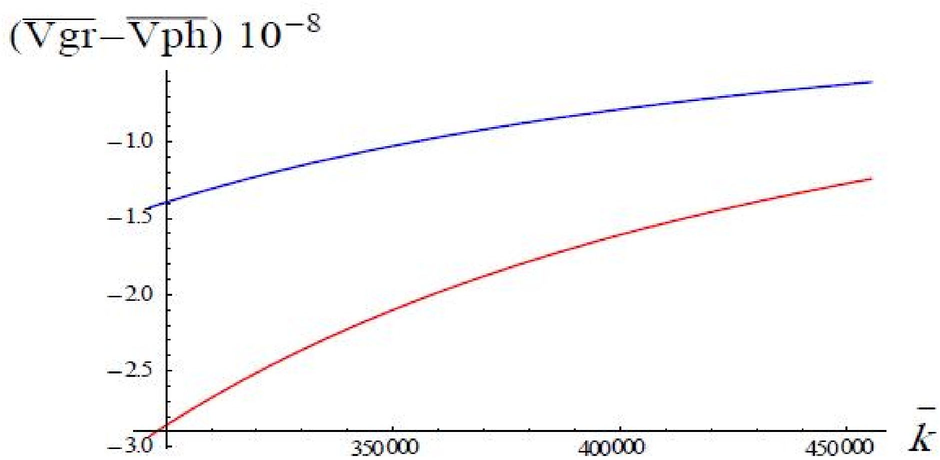


Fig. 1. Dispersion curves of ($\bar{V}_{gr}-\bar{V}_{ph}<0$) versus $\bar{k}$, the normal dispersion for PMMA and PS obeying ω_- , the two curves end points correspond to the laser wavelengths $\lambda=1.06\ 10^{-6}m$ and $\lambda=0.69\ 10^{-6}m$ respectively

PS(ω_-)

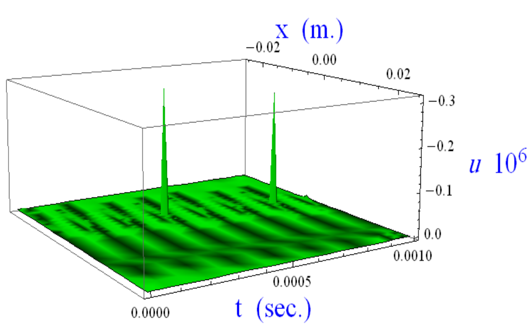


Fig. 2-a

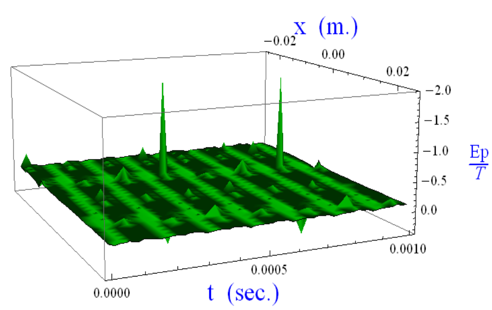


Fig. 2-b

Fig. 2 (a,b). The strain and the energy ratio of PS at the wave number k= 9.1×10^6 rad m^{-1}. and wave length λ=0.69μ

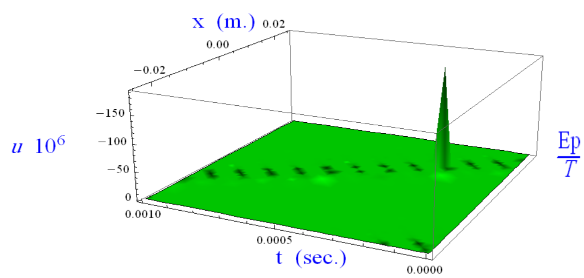
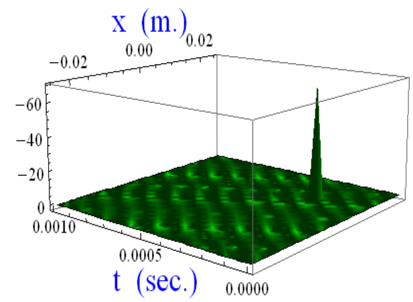


Fig (3-a)



Fig(3-b)

Fig. 3 (a,b). The strain and the energy ratio of PS at the wave number at the wave number k= 5.92753× 106 rad m^{-1}. and λ=1.06

PMMA(ω_-)

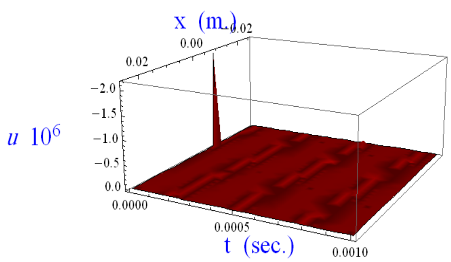


Fig. 4-a

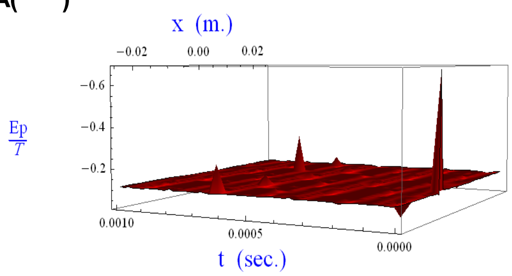


Fig. 4-b

Fig. 4 (a,b). The strain and the energy ratio of PMMA at the wave number k= 9.1×10^6 rad m^{-1}. and wave length λ=0.69μ

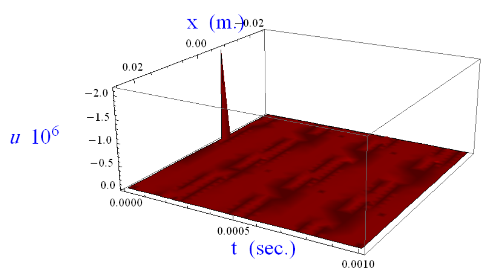


Fig. 5-a

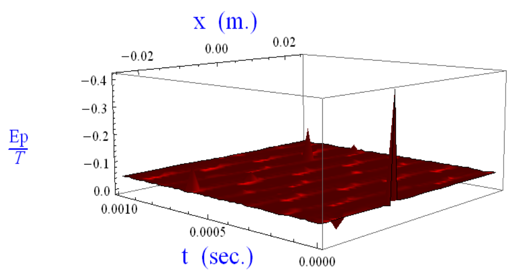


Fig. 5-b

Fig. 5 (a,b). The strain and the energy ratio of PMMA at the wave number k= 5.92753× 106 rad m^{-1}. and λ=1.06μ

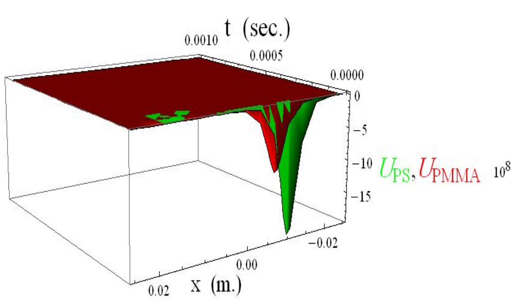


Fig. 6-a

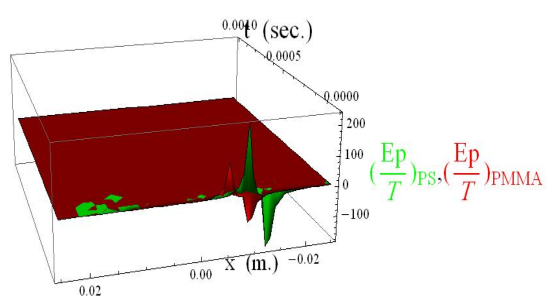


Fig. 6-b

Fig. 6 (a,b). The strain and the energy ratio of PMMA and PS conducted in [6]

The results are presented with respect to at the negative branch ω_-. It is found out that:

1- From Fig. 1, the difference between the group and phase velocities are illustrated for normal dispersion where $\bar{V}_{gr} - \bar{V}_{ph} = -6.4419 \times 10^{-8} < 0$ at λ=0.69 μ, and $\bar{V}_{gr} - \bar{V}_{ph} = -1.5203 \times 10^{-7} < 0$ at λ=1.06 μ, both velocities and wavenumbers are normalized with respect to the linear longitudinal wave velocity c_O and the specimen length *L=0.05 m*, for PMMA(in Red) and PS (in Blue) polymers respectively, see [19].

2- The Figs.(2-a,3-a,4-a,5-a) are representing the strain solitons stemmed from equations (12), together with the coefficients in (13), as a function of space x and time t. They are in excellent agreement with Figs.(6, a) attributed to the results in [15].

3- Figs.(2-b,3-b,4-b,5-b) correspond to the calculated ratio between the potential and kinetic energies in equations (14) and (15). It is found out that these ratios are fluctuating between +ve to -ve signs which is an acceptable behaviour due to laser ablation. In comparing these figures with Fig.(6, b), due to the data in Table (1) and the solution resulted in [15], good agreements with the experimental works in [9,10] take place.

6. CONCLUSION

In this article, we solve analytically the nonlinear Doubly Dispersive Equation (DDE) in (1+1)-D by the homogeneous balance method. The linear dispersion relation plays important role in connecting the frequency of the emitted nonlinear waves with the wave number of the ablating laser beam affecting the polymers with their characteristic parameters. The Tanh-function method is applied as a truncated series to estimate the nonlinear longitudinal bulk strain wave in a rod of complex polymer systems. In accordance with the normal dispersion conditions, the resulting solitary wave solutions show the compression characters in the nonlinearly elastic materials together with the fluctuating in signs presented by the ratio between the potential and kinetic energies due damage effects.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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