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The Solution of Caputo Fractional Partial Differential Equations By Sumudu Transform

D. S. Bodkhe [a*]

[a] *Department of Mathematics, Anandrao Dhonde Alias Babaji Mahavidyalaya, Kada, India.*

Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this paper, we obtained an explicit solutions of the fractional diffusion-wave equations involving partial Caputo fractional derivative by using Sumudu transform. The solutions of the fractional diffusion-wave equations obtained in terms of Mittag-Leffler function and generalized Wright function.Some illustrative examples are also given.

Keywords: Sumudu transform; wright function; mittag-Leffler function; fractional derivatives; fractional differential equations.

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**Corresponding author: E-mail: dsbodkhe@gmail.com;*

1 Introduction

In this paper, we apply the Sumudu transform to fractional integrals, derivatives, and use it to solve initial value fractional differential equations. In [1],[2],[3],[4],[5], [6], the authors studied many properties of the Sumudu transform in light of which they developed efficient and straightforward methodologies for treating ordinary [7] and partial differential equations. There is evident interest in further studying this transform, and applying it to various mathematical and physical sciences problems[8]. The Sumudu transform can be used to solve many types of difference and differential equations problems without resorting to a new frequency domain.The Sumudu transform was first defined by Watugala in 1993, which is used to solve engineering control problems [9], [10], [11]. The Weerakoon applied Sumudu transform to solve fractional differential equations [12],[13]. The fundamental properties of Sumudu transform are also used to solve the fractional differential equations [14],[15]. In this paper,we can find an explicit solution of the fractional diffusion-wave equations with Caputo fractional derivative by using the Sumudu transform method.

The fractional calculus is a generalization of differentiations and integrations to non-integers orders. We deal with the multiterm time-fractional partial differential equation involving the Caputo operator associated with the Laplace operator, which includes the momentum equations of the fractional Oldroyd-B fluid and the fractional Burgers fluid under some suitable conditions. By using several properties of multivariate Mittag–Leffler functions, the well-posedness and the long-time behavior for the Dirichlet problem are obtained. In addition, the uniqueness in inverse problem of determining orders of time-fractional derivatives of the equation is proved. There are many problems in physics and engineering formulated in terms of fractional differential and integral equations, such as diffusion, signal processing, electrochemistry, viscosity etc [18],[19]. The exact and approximate solutions of fractional differential equations are investigated by many authors using different methods. The Sumudu transform method is applied to obtain the solution of ordinary differential equations [20]. The Sumudu transform was first defined by Watugala in 1993, which is used to solve engineering control problems[9]. The Weerakoon applied Sumudu transform to solve fractional differential equations [12]. The fundamental properties of Sumudu transform are also used to solve the fractional differential equations [1],[14],[15] [16] [17]. In this paper, we can find an explicit solution of the fractional diffusion-wave equations with Caputo fractional derivative by using the Sumudu transform method.

2 Preliminary Results, Notations and Terminology

In this section we give definitions and some basic results which are used in the paper.
Consider the general linear fractional partial differential equation

Definition 2.1.

$$(D^{\alpha}_{o+,t}u)(x,t) = \sum_{j=1}^{n} a_j D^{\delta_j}_{x_j} u(x,t) + \sum_{j=1}^{n} b_j D^{\beta_j}_{x_j} u(x,t) + \sum_{j=1}^{n} c_j D^{\gamma_j}_{x_j} u(x,t) + du(x,t); \tag{2.1}$$

$$n-1 < \alpha \leq n, 2 < \delta_j \leq 3, 1 < \beta_j \leq 2, 0 < \gamma_j \leq 1, n \in \mathbb{N}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, a_j, b_j c_j, d$ *are non-negative real constants,* $o \leq t < T$

The fractional diffusion equation is

$$(D_{o+,t}u)(x,t) = \sum_{j=1}^{n} b_j D^{\beta_j}_{x_j} u(x,t) \tag{2.2}$$

$n-1 < \alpha \leq n, 1 < \beta_j \leq 2, n \in \mathbb{N}$

Definition 2.2. *[21]The Riemann-Liouville fractional integral of order* $\alpha, \alpha > 0$ *of a function* $u(x,t)$ *is denoted by* $I^{\alpha}_{0+,t}u(x,t)$ *and defined as*

$$I^{\alpha}_{0+,t}u(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(x,\tau) d\tau, \quad t > 0, \alpha > 0 \tag{2.3}$$

Definition 2.3. *[21]The Caputo fractional derivative of order $\alpha, \alpha > 0$ of a function $u(x,t)$ is denoted by ${}^{c}D^{\alpha}_{0+,t}u(x,t)$ and defined as*

$$ {}^{c}D^{\alpha}_{0+,t}, u(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{\frac{\partial^n}{\partial t^n} u(x,\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad (x \in \mathbb{R}, t > 0, n-1 < \alpha < n, n \in \mathbb{N}) \tag{2.4} $$

2.1 Mittag-Leffler Function

The Mittag-Leffler function was introduced by M. G. Mittag-Leffler and is denoted by $E_{\alpha}(z)$ [22]. It is one parameter generalization of exponential function and is defined as,

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \ \alpha \in C, \ Re(\alpha) > 0. \tag{2.5} $$

A two-parameter Mittag-Leffler function introduced by R. P. Agarwal [3], denoted by $E_{\alpha,\,\beta}(z)$, is defined as,

$$ E_{\alpha,\,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ \alpha > 0, \ \beta > 0. \tag{2.6} $$

2.2 Wright Function

The Wright function introduced by Wright, denoted by $\phi(\alpha, \beta; z)$ [21] is a generalization of Mittag-Leffler function and is defined as,

$$ \phi(\alpha, \beta; z) = {}_0\Psi_1 \left[\begin{array}{c|c} ------ & z \\ (\beta, \alpha) & \end{array} \right] = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!} \tag{2.7} $$

The more general Wright function ${}_p\Psi_q(z)$ $z, a_l, b_j \in \mathbb{C}$ and $\alpha_l, \beta_j \in \mathbb{R}$ $(l = 1, 2, \ldots, p \ ; \ j = 1, 2, \ldots, q)$ is defined by the series

$$ {}_p\Psi_q(z) = {}_1\Psi_1 \left[\begin{array}{c|c} (a_l,\, \alpha_l)_{1,p} & z \\ (b_l,\, \beta_l)_{1,q} & \end{array} \right] = \sum_{k=0}^{\infty} \frac{\Pi_{l=1}^{p} \Gamma(a_l + \alpha_l k)}{\Pi_{j=1}^{q} \Gamma(b_j + \beta_j k)} \cdot \frac{z^k}{k!} \tag{2.8} $$

The Wright function with $p = q = 1$ of the form

$$ {}_1\Psi_1 \left[\begin{array}{c|c} (n+1,\, n) & z \\ (\alpha n + \beta,\, \alpha) & \end{array} \right] = \sum_{j=0}^{\infty} \frac{\Gamma(n + j + 1)}{\Gamma(\alpha n + \beta + \alpha j)} \cdot \frac{z^j}{j!} = \left(\frac{\partial}{\partial z} \right)^n E_{\alpha,\,\beta}(z). \tag{2.9} $$

Consider a set A defined as [9]

$$ A = \{ f(t) |\ \exists\, M, \tau_1, \tau_2 > 0, |f(t)| \leq M e^{\frac{|t|}{\tau_j}} \ \text{if } t \in (-1)^j \times [0, \infty) \}. $$

Definition 2.4. *For all real $t \geq 0$, the Sumudu transform of a function $\in A$, with respect to t denoted by $(S_t u)(x,p)$, is defined as*

$$ (S_t u)(x,p) = \int_0^{\infty} u(x,t) p e^{pt} dt, \quad u \in (-\tau_1, \tau_2), \quad \textit{where} \left(p = -\frac{1}{u} \right) \qquad (x \in \mathbb{R}; p > 0) \tag{2.10} $$

and the inverse Sumidu transform with respect to p is

$$ (S_p^{-1} u)(x,p) = \frac{1}{2\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt} u(x,p) dp, \quad (x \in \mathbb{R}; \gamma = \Re(p) > \sigma u). \tag{2.11} $$

[8]The Sumudu transform of the Caputo fractional derivative of order $\alpha, \alpha > 0$ of a function $u(x,p)$ is denoted by $(S_t^c D^{\alpha}_{0+,t} u)(x,p)$ and defined as

$$ (S_t^c D^{\alpha}_{0+,t} u)(x,p) = p^{\alpha} (Su)(x,p) - \sum_{k=0}^{l-1} p^{\alpha - j - 1} \left(\frac{\partial^k u(x,0)}{\partial t^k} \right) \tag{2.12} $$

with $x \in \mathbb{R}, l - 1 < \alpha \leq l$ and $n \in \mathbb{N}$.

Definition 2.5. *[21]The Fourier transform with respect to $x \in \mathbb{R}$ denoted by $(F_x u)(\sigma, t)$ is defined as*

$$(F_x u)(\sigma, t) = \int_{-\infty}^{\infty} u(x,t) e^{ix\sigma} dx, \quad (\sigma \in \mathbb{R}; t > 0) \tag{2.13}$$

and the inverse of the Fourier transform of $u(x,t)$ with respect to σ is

$$(F_\sigma^{-1} u)(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(\sigma, t) e^{-i\sigma x} d\sigma, \quad (\sigma \in \mathbb{R}; t > 0) \tag{2.14}$$

the relation with respect to $x \in \mathbb{R}$

$$F[D^k \phi(t)](x) = (-ix)^k (F\phi) \quad (k \in \mathbb{N}) \tag{2.15}$$

and the Fourier convolution operator of two functions f and g is defined by the integral

$$f * g = (f * g)(x) = \int_{\infty}^{\infty} f(x-t) g(t) dt, \quad (x \in \mathbb{R}) \tag{2.16}$$

3 Solution of Cauchy Type Problems for Fractional Diffusion-Wave Equations With Caputo derivatie

In this section we apply the Sumudu transform method to the fractional diffusion-wave equations with Caputo fractional derivative. we can derive the explicit solution to the fractional diffusion-wave equations of the form In this section we consider a fractional differential equation of the form

$$({}^c D_{0+,t}^{\alpha} u)(x,t) = \lambda (\triangle_x u)(x,t) \quad (x \in \mathbb{R}; t > 0; 0 < \alpha < 2; \lambda > 0) \tag{3.1}$$

involving the partial Caputo fractional derivative $({}^c D_{0+,t}^{\alpha} u)(x,t)$ with respect to $t > 0$,and the Sumudu $(\triangle_x u)(x,t)$ with respect to $x \in \mathbb{R}^n$ given by

$$(\triangle_x u)(x,t) = \frac{\partial^2 u(x,t)}{\partial {x_1}^2} + \ldots + \frac{\partial^2 u(x,t)}{\partial {x_n}^2} \quad (n \in \mathbb{N}) \tag{3.2}$$

In definition Caputo fractional derivative (2.4) of order $\alpha > 0$ and $l - 1 < \alpha \leq l (l \in \mathbb{N})$ is defined in terms of the Riemann-Liouville partial fractional derivative $eqref cdrl$ by

$$({}^c D_{0+,t}^{\alpha} u)(x,t) = \left(D_{0+,t}^{\alpha} \left[u(x,\tau) - \sum_{k=0}^{l-1} \frac{\partial^k u(x,0)}{\partial t^k} \frac{\tau^k}{k!} \right] \right)(x,t) \tag{3.3}$$

When, for any fixed $x \in \mathbb{R}, u(x,t) \in C^l(\mathbb{R}_+)$ as a function of $t > 0$, then $({}^c D_{0+,t}^{\alpha} u)(x,t)$ has the representation (2.4). In this section we apply the Fourier and Sumudu transform to obtain an explicit solution to the equation (3.1) with Cauchy initial conditions

$$\frac{\partial^k u(x,0)}{\partial t^k} = f_k(x) \quad . \ (x \in \mathbb{R}; k = 0 \quad \text{for} \quad 0 < \alpha \leq 1; k = 1 \quad \text{for} \quad 1 < \alpha < 2). \tag{3.4}$$

$$\frac{\partial^0 u(x,0)}{\partial t^0} = u(x,0) \tag{3.5}$$

Example 3.1. *Solve the following partial differential equation of order $\alpha, 0 < \alpha < 2$*

$$({}^c D_{0+,t}^{\alpha} u)(x,t) = \lambda^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad (x \in \mathbb{R}; t > 0; 0 < \alpha < 2; \lambda > 0). \tag{3.6}$$

$$\frac{\partial^k u(x,0)}{\partial t^k} = f_k(x) \quad . \ (x \in \mathbb{R}; k = 0 \quad for \quad 0 < \alpha \leq 1; k = 1 \quad for \quad 1 < \alpha < 2). \tag{3.7}$$

Applying the Sumudu transform (2.10) to equation (3.6) using the initial condition (3.7) with respect t

$$S[({}^{c}D^{\alpha}_{0+,t}u)(x,p)] = \lambda^2\left(\frac{\partial^2}{\partial x^2}S_t u\right)(x,p) \quad (l=1,2) \tag{3.8}$$

Using (3.13) with $x \in \mathbb{R}, l-1 < \alpha \leq l$ and $n \in \mathbb{N}$. If $l=1$ and $l=2$ in respective cases $0 < \alpha \leq 1$ and $1 < \alpha < 2$,and the initial conditions in (3.7), we have

$$p^{\alpha}(S_t u)(x,p) = \sum_{k=0}^{l-1} p^{\alpha-k-1}f_k(x) + \lambda^2\left(\frac{\partial^k}{\partial x^k}\right) \quad (l=1,2) \tag{3.9}$$

Now, applying the Fourier transform (2.13) and using the formula (2.15) with $k=2$, we have

$$\left(F_x[\frac{\partial^2 u(x,t)}{\partial x^2}]\right)(\sigma,t) = -|\sigma|^2(F_x u)(\sigma,t), \tag{3.10}$$

By applying Fourier transform on equation (3.9) and using equation (3.10), we obtain

$$p^{\alpha}(F_x S_t u)(\sigma,p) = \sum_{k=0}^{l-1} p^{\alpha-k-1}(F_x f_k)(\sigma) - \lambda^2|\sigma|^2(F_x S_t u)(\sigma,p)$$

$$(F_x S_t u)(\sigma,p) = \sum_{k=0}^{l1} \frac{p^{\alpha-k-1}}{p^{\alpha}+\lambda^2|\sigma|^2}(F_x f_k)(\sigma),, \quad (\sigma \in \mathbb{R}, t>0, l=1,2). \tag{3.11}$$

Now, we obtain the explicit solution $u(x,t)$ by using the inverse Fourier transform (2.14) with respect to σ and the inverse Sumudu transform (2.11) with respect to p.

$$(F_x e^{-c|x|})(\sigma) = \frac{2c}{c^2+|\sigma|^2} \quad (\sigma \in \mathbb{R}; c>0)$$

and

$$(Fe^{-\frac{|x|}{\lambda}p^{\frac{\alpha}{2}}}) = \frac{2\lambda p^{\frac{\alpha}{2}}}{p^{\alpha}+\lambda^2|\sigma|^2}$$

From these equations, we have

$$\left(F_x\big[\frac{1}{2\lambda}p^{\frac{\alpha}{2}-k-1}e^{-\frac{|x|}{\lambda}p^{\frac{\alpha}{2}}}\big]\right) = \frac{p^{\alpha-k-1}}{p^{\alpha}+\lambda^2|\sigma|^2}(\sigma) \quad (k=0,1)$$

Hence equation (3.11) becomes

$$(F_x S_t u)(\sigma,p) = \left(F_x\big[\frac{1}{2\lambda}p^{\frac{\alpha}{2}-k-1}e^{-\frac{|x|}{\lambda}p^{\frac{\alpha}{2}}}\big]\right)(\sigma)(F_x f_k)(\sigma) \qquad (l=1,2).$$

By using the Fourier transform convolution property, we have

$$(F_x S_t u)(\sigma,p) = \left(F_x\big[\frac{1}{2\lambda}p^{\frac{\alpha}{2}-k-1}e^{-\frac{|x|}{\lambda}p^{\frac{\alpha}{2}}} * x f_k\big]\right)(\sigma) \qquad (l=1,2).$$

Now applying the inverse Fourier transform (2.14), we obtain

$$(S_t u)(x,p) = \frac{1}{2\lambda}p^{\frac{\alpha}{2}-k-1}e^{-\frac{|x|}{\lambda}p^{\frac{\alpha}{2}}} * x f_k \qquad (l=1,2). \tag{3.12}$$

When $0 < \alpha < 2$, then the function $p^{\frac{\alpha}{2}-k-1}e^{-\frac{|x|}{\lambda}p^{\frac{\alpha}{2}}}(k=0,1)$ are expressed through sumudu transform of the Wright function $\phi(-\frac{\alpha}{2}, b; z)$ as follows.

$$\left(S_t\left[t^{\frac{\alpha}{2}-k}\phi(-\frac{\alpha}{2}; k+1-\frac{\alpha}{2}; -\frac{|x|}{\lambda}t^{-\frac{\alpha}{2}})\right]\right)(p) = p^{\frac{\alpha}{2}-k-1}e^{-\frac{|x|}{\lambda}p^{\frac{\alpha}{2}}} \quad \text{for}(k=0,1) \tag{3.13}$$

Applying the inverse Sumudu transform to (2.11) and using (3.13), we can obtain the solution.

$$u(x,t) = \frac{1}{2\lambda}t^{\frac{\alpha}{2}-k}\phi(-\frac{\alpha}{2}; k+1-\frac{\alpha}{2}; -\frac{|x|}{\lambda}t^{-\frac{\alpha}{2}}) \quad \text{for}(k=0,1)$$

Theorem 3.1. *If* $0 < \alpha < 2$ *and* $\lambda > 0$*, then the Cauchy type problem* (3.6) *and* (3.7) *is solvable and its solution is given by*

$$u(x,t) = \sum_{k=1}^{l} \int_{-\infty}^{\infty} G_k^{\alpha}(x-\tau,t) f_k(\tau) d\tau \tag{3.14}$$

$$(l = 1 \quad for \quad 0 < \alpha \leq 1; l = 2 \quad for \quad 1 < \alpha < 2)$$

where

$$G_k^{\alpha}(x,t) = \frac{1}{2\lambda} t^{k-\frac{\alpha}{2}} \phi\bigg(-\frac{\alpha}{2}; k+1-\frac{\alpha}{2}; -\frac{|x|}{\lambda} t^{-\frac{\alpha}{2}}\bigg) \quad for \quad (k = 0,1)$$

provided that integral in the right-hand side of (3.14) are convergent.

Proof: Applying the Sumudu transform (2.10) to equation (3.6) using the initial condition (3.7) with respect t

$$S[({}^{c}D_{0+,t}^{\alpha} u)(x,p)] = \lambda^2 \bigg(\frac{\partial^2}{\partial x^2} S_t u\bigg)(x,p) \quad (l = 1,2) \tag{3.15}$$

Using (3.13) with $x \in \mathbb{R}, l-1 < \alpha \leq l$ and $n \in \mathbb{N}$. If $l = 1$ and $l = 2$ in respective cases $0 < \alpha \leq 1$ and $1 < \alpha < 2$, and the initial conditions in (3.7), we have

$$p^{\alpha}(S_t u)(x,p) = \sum_{k=0}^{l-1} p^{\alpha-k-1} f_k(x) + \lambda^2 \bigg(\frac{\partial^k}{\partial x^k}\bigg) \quad (l = 1,2) \tag{3.16}$$

Now, applying the Fourier transform (2.13) and using the formula (2.15) with $k = 2$, we have

$$\bigg(F_x[\frac{\partial^2 u(x,t)}{\partial x^2}]\bigg)(\sigma,t) = -|\sigma|^2 (F_x u)(\sigma,t), \tag{3.17}$$

By applying Fourier transform on equation (2.13) and using equation (3.17), we obtain

$$p^{\alpha}(F_x S_t u)(\sigma,p) = \sum_{k=0}^{l-1} p^{\alpha-k-1} (F_x f_k)(\sigma) - \lambda^2 |\sigma|^2 (F_x S_t u)(\sigma,p)$$

$$(F_x S_t u)(\sigma,p) = \sum_{k=0}^{l1} \frac{p^{\alpha-k-1}}{p^{\alpha} + \lambda^2 |\sigma|^2} (F_x f_k)(\sigma), \quad (\sigma \in \mathbb{R}, t > 0, l = 1,2). \tag{3.18}$$

Now, we obtain the explicit solution $u(x,t)$ by using the inverse Fourier transform (2.14) with respect to σ and the inverse Sumudu transform (2.11) with respect to p.

$$(F_x e^{-c|x|})(\sigma) = \frac{2c}{c^2 + |\sigma|^2} \quad (\sigma \in \mathbb{R}; c > 0)$$

and

$$(Fe^{-\frac{|x|}{\lambda} p^{\frac{\alpha}{2}}}) = \frac{2\lambda p^{\frac{\alpha}{2}}}{p^{\alpha} + \lambda^2 |\sigma|^2}$$

From these equations, we have

$$\bigg(F_x\big[\frac{1}{2\lambda} p^{\frac{\alpha}{2}-k-1} e^{-\frac{|x|}{\lambda} p^{\frac{\alpha}{2}}}\big]\bigg) = \frac{p^{\alpha-k-1}}{p^{\alpha} + \lambda^2 |\sigma|^2}(\sigma) \quad (k = 0,1)$$

Hence, equation (3.18) becomes

$$(F_x S_t u)(\sigma,p) = \bigg(F_x\big[\frac{1}{2\lambda} p^{\frac{\alpha}{2}-k-1} e^{-\frac{|x|}{\lambda} p^{\frac{\alpha}{2}}}\big]\bigg)(\sigma)(F_x f_k)(\sigma) \qquad (l = 1,2).$$

By using the Fourier transform convolution property (2.16) we have

$$(F_x S_t u)(\sigma, p) = \left(F_x\big[\frac{1}{2\lambda}p^{\frac{\alpha}{2}-k-1}e^{-\frac{|x|}{\lambda}p^{\frac{\alpha}{2}}} * xf_k\big]\right)(\sigma) \qquad (l=1,2).$$

Now applying the inverse Fourier transform (2.14), we obtain

$$(S_t u)(x,p) = \frac{1}{2\lambda}p^{\frac{\alpha}{2}-k-1}e^{-\frac{|x|}{\lambda}p^{\frac{\alpha}{2}}} * xf_k \qquad (l=1,2). \tag{3.19}$$

$$u(x,t) = \sum_{k=1}^{l}\int_{-\infty}^{\infty} G_k^{\alpha}(x-\tau,t)f_k(\tau)d\tau$$
$$(l=1 \quad for \quad 0<\alpha\le 1; l=2 \quad for \quad 1<\alpha<2)$$

where $G^{\alpha}(x,t)$ is the green function can be written as

$$G^{\alpha}(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\sum_{k=0}^{l-1}\frac{1}{2\lambda}p^{\frac{\alpha}{2}-k-1}e^{-\frac{|x|}{\lambda}p^{\frac{\alpha}{2}}}xf_k(x)dp \qquad (l=1,2). \tag{3.20}$$

When $0<\alpha<2$, then the function $p^{\frac{\alpha}{2}-k-1}e^{-\frac{|x|}{\lambda}p^{\frac{\alpha}{2}}}(k=0,1)$ are expressed through sumudu transform of the Wright function $\phi(-\frac{\alpha}{2},b;z)$ as follows.

$$\left(S_t\left[t^{\frac{\alpha}{2}-k}\phi(-\frac{\alpha}{2};k+1-\frac{\alpha}{2};-\frac{|x|}{\lambda}t^{-\frac{\alpha}{2}})\right]\right)(p) = p^{\frac{\alpha}{2}-k-1}e^{-\frac{|x|}{\lambda}p^{\frac{\alpha}{2}}} \quad \text{for}(k=0,1) \tag{3.21}$$

Applying the inverse Sumudu transform to (3.19) and using (3.21), we can obtain the solutin.

$$u(x,t) = \frac{1}{2\lambda}t^{\frac{\alpha}{2}-k}\phi(-\frac{\alpha}{2};k+1-\frac{\alpha}{2};-\frac{|x|}{\lambda}t^{-\frac{\alpha}{2}}) \quad \text{for} \quad (k=0,1)$$

Corollary 3.1. *If $0<\alpha<1$ and $\lambda>0$, then the Cauchy problem*

$$({}^cD_{0+,t}^{\alpha}u)(x,t) = \lambda^2\frac{\partial^2 u(x,t)}{\partial x^2} \quad (x\in\mathbb{R}; t>0)$$
$$u(x,0) = f(x) \tag{3.22}$$

is solvable, and its solution has the form

$$u(x,t) = \int_{-\infty}^{\infty} G_1^{\alpha}(x-\tau,t)f_1(\tau)d\tau \tag{3.23}$$

where

$$G_1^{\alpha}(x,t) = \frac{1}{2\lambda}t^{-\frac{\alpha}{2}}\phi\left(-\frac{\alpha}{2},1-\frac{\alpha}{2};-\frac{|x|}{\lambda}t^{-\frac{\alpha}{2}}\right) \tag{3.24}$$

provided that the integral in the right-hand side of (3.23) is convergent.

Corollary 3.2. *If $0<\alpha<2$ and $\lambda>0$, then the Cauchy problem*

$$({}^cD_{0+,t}^{\alpha}u)(x,t) = \lambda^2\frac{\partial^2 u(x,t)}{\partial x^2}, \quad (x\in\mathbb{R}; t>0) \tag{3.25}$$

$$u(x,0) = f_0(x)$$
$$\frac{\partial u(x,0)}{\partial t} = f_1(x) \quad (x\in\mathbb{R}) \tag{3.26}$$

is solvable, and its solution has the form

$$u(x,t) = \int_{-\infty}^{\infty} G_1^{\alpha}(x-\tau,t) f_0(\tau) d\tau + \int_{-\infty}^{\infty} G_2^{\alpha}(x-\tau,t) f_1(\tau) d\tau \tag{3.27}$$

where

$$G_1^{\alpha}(x,t) = \frac{1}{2\lambda} t^{-\frac{\alpha}{2}} \phi\bigg(-\frac{\alpha}{2}, 1-\frac{\alpha}{2}; -\frac{|x|}{\lambda} t^{-\frac{\alpha}{2}}\bigg) \tag{3.28}$$

and

$$G_2^{\alpha}(x,t) = \frac{1}{2\lambda} t^{1-\frac{\alpha}{2}} \phi\bigg(-\frac{\alpha}{2}, 2-\frac{\alpha}{2}; -\frac{|x|}{\lambda} t^{-\frac{\alpha}{2}}\bigg) \tag{3.29}$$

provided that the integral in the right-hand side of (3.27) is convergent.

Example 3.2. *Solve the following Cauchy problem with* $\alpha = \frac{1}{2}$

$$\begin{aligned} ({}^{c}D_{0+,t}^{\frac{1}{2}} u)(x,t) &= \lambda^2 \frac{\partial^2 u(x,t)}{\partial x^2} \quad (x \in \mathbb{R}; t > 0) \\ u(x,0) &= f(x) \end{aligned} \tag{3.30}$$

is solvable, and its solution has the form

$$u(x,t) = \int_{-\infty}^{\infty} G_1^{\frac{1}{2}}(x-\tau,t) f_1(\tau) d\tau \tag{3.31}$$

where

$$G_1^{\frac{1}{2}}(x,t) = \frac{1}{2\lambda} t^{-\frac{1}{4}} \phi\bigg(-\frac{1}{4}, \frac{3}{4}; -\frac{|x|}{\lambda} t^{-\frac{1}{4}}\bigg) \tag{3.32}$$

Applying the Sumudu transform (2.10) to equation (3.30) and using the initial condition with respect t

$$S[({}^{c}D_{0+,t}^{\frac{1}{2}} u)(x,p)] = \lambda^2 \bigg(\frac{\partial^2}{\partial x^2} S_t u\bigg)(x,p) \quad (l=1) \tag{3.33}$$

Using (2.12) with $x \in \mathbb{R}, l-1 < \alpha \leq l$ and $n \in \mathbb{N}$. If $l = 1$ in respective cases $0 < \alpha \leq 1$ and the initial conditions in (3.30), we have

$$p^{\frac{1}{2}} (S_t u)(x,p) = p^{-\frac{1}{2}} f_0(x) + \lambda^2 \bigg(\frac{\partial^k}{\partial x^k}\bigg) \quad (l=1) \tag{3.34}$$

Now, applying the Fourier transform (2.13)to equation (3.30) and using the formula (2.15) with $k = 2$, we have

$$\bigg(F_x[\frac{\partial^2 u(x,t)}{\partial x^2}]\bigg)(\sigma,t) = -|\sigma|^2 (F_x u)(\sigma,t) \tag{3.35}$$

By applying Fourier transform on equation(3.34) and using equation(3.35), we obtain

$$\begin{aligned} p^{\frac{1}{2}} (F_x S_t u)(\sigma,p) &= p^{-\frac{1}{2}} f_0(x)(\sigma) - \lambda^2 |\sigma|^2 (F_x S_t u)(\sigma,p), \quad (\sigma \in \mathbb{R}, t > 0, l = 1,2) \\ (F_x S_t u)(\sigma,p) &= \frac{p^{-\frac{1}{2}} f_0(x)}{p^{\frac{1}{2}} + \lambda^2 |\sigma|^2} (F_x f_k)(\sigma), \quad (\sigma \in \mathbb{R}, t > 0, l = 1). \end{aligned} \tag{3.36}$$

The Fourier transform are

$$(F_x e^{-c|x|})(\sigma) = \frac{2c}{c^2 + |\sigma|^2} \quad (\sigma \in \mathbb{R}; c > 0)$$

and

$$(F e^{-\frac{|x|}{\lambda} p^{\frac{1}{4}}}) = \frac{2\lambda p^{\frac{1}{4}}}{p^{\frac{1}{2}} + \lambda^2 |\sigma|^2}$$

Hence equation (3.36) becomes

$$(F_x S_t u)(\sigma,p) = \left(F_x[\frac{1}{2\lambda}p^-\frac{3}{4}e^{-\frac{|x|}{\lambda}}p^{\frac{1}{4}}]\right)(\sigma)(F_x f_0)(\sigma) \qquad (l=1)$$

By the convolution property of Fourier transform (2.16) we have

$$(F_x S_t u)(\sigma,p) = \left(F_x[\frac{1}{2\lambda}p^-\frac{3}{4}e^{-\frac{|x|}{\lambda}}p^{\frac{1}{4}}xf_0]\right)(\sigma) \qquad (l=1)$$

Now applying the inverse Fourier transform, we obtain

$$(S_t u)(\sigma,p) = \left([\frac{1}{2\lambda}p^-\frac{3}{4}e^{-\frac{|x|}{\lambda}}p^{\frac{1}{4}}xf_0]\right)(\sigma) \qquad (l=1) \tag{3.37}$$

Thus, we get

$$u(x,t) = \int_{-\infty}^{\infty} G_1^{\frac{1}{2}}(x-\tau,t)f_1(\tau)d\tau$$

where $G_1^{\frac{1}{2}}(x,t)$ is the Green function can be written as

$$G_1^{\frac{1}{2}}(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{2\lambda}p^{-\frac{1}{4}}e^{-}\frac{|x|}{\lambda}p^{\frac{1}{4}} * xf(x)dp \tag{3.38}$$

To obtain the solution of $G_1^{\frac{1}{2}}(x,t)$, when $0<\alpha\leq 1$ then the function $p^-\frac{3}{4}e^{-\frac{|x|}{\lambda}}p^{\frac{1}{4}}$ can be expressed through Sumudu transform of the Wright function $\phi(-\frac{1}{4},b;z)$ as follows

$$\left(S_t[t^-\frac{1}{4}\phi(-\frac{1}{4},\frac{3}{4};-\frac{|x|}{\lambda}p^-\frac{1}{4})]\right)(p) = p^-\frac{3}{4}e^{-\frac{|x|}{\lambda}}p^{\frac{1}{4}} \qquad (k=0) \tag{3.39}$$

Applying the inverse Sumudu transform to (3.37) and using (3.38), we get where

$$G_1^{\frac{1}{2}}(x,t) = \frac{1}{2\lambda}t^{-\frac{1}{4}}\phi\left(-\frac{1}{4};\frac{3}{4};-\frac{|x|}{\lambda}t^{-\frac{1}{4}}\right) \quad \text{for} \quad (k=1)$$

Example 3.3. *Solve the following Cauchy problem with* $\alpha=\frac{3}{2}$

$$({}^cD_{0+,t}^{\frac{3}{2}}u)(x,t) = \lambda^2\frac{\partial^2 u(x,t)}{\partial x^2} \quad (x\in\mathbb{R};t>0) \tag{3.40}$$

$$\begin{aligned} u(x,0) &= f_0(x) \ \ . \ (x\in\mathbb{R}) \\ \frac{\partial u(x,0)}{\partial t} &= f_1(x) \ \ . \ (x\in\mathbb{R}). \end{aligned} \tag{3.41}$$

has its solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} G_1^{\frac{3}{2}}(x-\tau,t)f_0(\tau)d\tau + \int_{-\infty}^{\infty} G_2^{\frac{3}{2}}(x-\tau,t)f_1(\tau)d\tau \tag{3.42}$$

where

$$G_1^{\frac{3}{2}}(x,t) = \frac{1}{2\lambda}t^{-\frac{3}{4}}\phi\left(-\frac{3}{4};\frac{1}{4};-\frac{|x|}{\lambda}t^{-\frac{3}{4}}\right) \tag{3.43}$$

and

$$G_2^{\frac{3}{2}}(x,t) = \frac{1}{2\lambda}t^{\frac{1}{4}}\phi\left(-\frac{3}{4};\frac{5}{4};-\frac{|x|}{\lambda}t^{-\frac{3}{4}}\right) \tag{3.44}$$

Applying Sumudu transform (2.10) to (3.40) and using the initial condition (3.41), with respect to t, we obtain

$$S[(D^{\alpha}_{0+,t}u)(x,p)] = \lambda^2\left(\frac{\partial^2}{\partial x^2}S_t u\right)(x,p) \quad (l = 1,2) \tag{3.45}$$

Using (2.12) with $x \in \mathbb{R}, l-1 < \alpha \leq l$ and $n \in \mathbb{N}$. If $l = 1,2$ in respective cases $0 < \alpha \leq 1$ and$1 < \alpha < 2$ the initial conditions in (3.41), we have

$$p^{\frac{3}{2}}(S_t u)(x,p) = p^{\frac{1}{2}}f_0(x) - p^{-\frac{1}{2}}f_1(x) + \lambda^2\left(\frac{\partial^2 u(x,0)}{\partial x^2}\right) \quad (l = 1,2) \tag{3.46}$$

Now, applying the Fourier transform (2.13) to equation (3.46) and using the formula (2.15) with $k = 2$, we have

$$\left(F_x\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right]\right)(\sigma,t) = -|\sigma|^2(F_x u)(\sigma,t), \tag{3.47}$$

By applying Fourier transform on equation (3.46) and using equation(3.47), we obtain

$$p^{\frac{3}{2}}(F_x S_t u)(\sigma,p) = p^{\frac{1}{2}}F_x f_0(x)(\sigma) + p^{-\frac{1}{2}}F_x f_1(x)(\sigma) - \lambda^2|\sigma|^2(F_x S_t u)(\sigma,p)$$

$$(F_x S_t u)(\sigma,p) = \frac{p^{\frac{1}{2}}f_0(x)}{p^{\frac{3}{2}} + \lambda^2|\sigma|^2} + \frac{p^{-\frac{1}{2}}f_0(x)}{p^{\frac{3}{2}} + \lambda^2|\sigma|^2}(F_x f_k)(\sigma), \;\; (\sigma \in \mathbb{R}, t > 0, l = 1,2). \tag{3.48}$$

The Fourier transform are

$$(F_x e^{-c|x|})(\sigma) = \frac{2c}{c^2 + |\sigma|^2} \quad (\sigma \in \mathbb{R}; c > 0)$$

and

$$(Fe^{-\frac{|x|}{\lambda}p^{\frac{3}{4}}}) = \frac{2\lambda p^{\frac{3}{4}}}{p^{\frac{3}{2}} + \lambda^2|\sigma|^2}$$

Hence equation (3.48) becomes

$$(F_x S_t u)(\sigma,p) = \left(F_x[\frac{1}{2\lambda}p^{\frac{-3}{4}}e^{\frac{-|x|}{\lambda}}p^{\frac{1}{4}}]\right)(\sigma)(F_x f_0)(\sigma) \qquad (l = 1,2)$$

By the convolution property of Fourier transform (2.16) we have

$$(F_x S_t u)(\sigma,p) = \left(F_x[\frac{1}{2\lambda}p^{-}\frac{1}{4}e^{-\frac{|x|}{\lambda}}p^{\frac{3}{4}}xf_0]\right)(\sigma) + \left(F_x[\frac{1}{2\lambda}p^{-}\frac{5}{4}e^{-\frac{|x|}{\lambda}}p^{\frac{3}{4}}xf_1]\right)(\sigma) \quad (l = 1,2).$$

Now applying the inverse Fourier transform,we obtain

$$(S_t u)(x,p) = \left([\frac{1}{2\lambda}p^{-}\frac{1}{4}e^{-\frac{|x|}{\lambda}}p^{\frac{3}{4}}xf_0(x)]\right) + \left([\frac{1}{2\lambda}p^{-}\frac{5}{4}e^{-\frac{|x|}{\lambda}}p^{\frac{3}{4}}xf_1(x)]\right) \qquad (l = 1,2) \tag{3.49}$$

Thus we get

$$u(x,t) = \int_{-\infty}^{\infty} G_1^{\frac{3}{2}}(x-\tau,t)f_0(\tau)d\tau + \int_{-\infty}^{\infty} G_2^{\frac{3}{2}}(x-\tau,t)f_1(\tau)d\tau$$

Where $G_1^{\frac{3}{2}}(x,t)$ and $G_2^{\frac{3}{2}}(x,t)$ are the Green's functions can be written as

$$G_1^{\frac{3}{2}}(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{2\lambda}p^{-\frac{1}{4}}e^{-\frac{|x|}{\lambda}p^{\frac{3}{4}}} * xf(x)dp \tag{3.50}$$

and

$$G^{\frac{3}{2}}(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{2\lambda}p^{-\frac{5}{4}}e^{-\frac{|x|}{\lambda}p^{\frac{3}{4}}} * xf(x)dp \tag{3.51}$$

To obtain the solution of $G_1^{\frac{3}{2}}(x,t)$ and $G_2^{\frac{3}{2}}(x,t)$ when $0<\alpha\leq 1$ and $1<\alpha<2$ then the function $p^{-}\frac{1}{4}e^{-\frac{|x|}{\lambda}}p^{\frac{3}{4}}$ and $p^{-}\frac{5}{4}e^{-\frac{|x|}{\lambda}}p^{\frac{3}{4}}$ can be expressed through Sumudu transform of the Wright function $\phi(-\frac{3}{4},b;z)$ and $\phi(-\frac{3}{4},b;z)$ as follows

$$\begin{aligned}\Big(S_t[t^{-}\frac{3}{4}\phi(-\frac{3}{4},\frac{1}{4};-\frac{|x|}{\lambda}p^{-}\frac{3}{4})]\Big)(p)+\Big(S_t[t^{\frac{1}{4}}\phi(-\frac{3}{4},\frac{5}{4};-\frac{|x|}{\lambda}p^{-}\frac{3}{4})]\Big)(p)\\ =p^{-}\frac{1}{4}e^{-\frac{|x|}{\lambda}}p^{\frac{3}{4}}+p^{-}\frac{5}{4}e^{-\frac{|x|}{\lambda}}p^{\frac{3}{4}}\qquad(k=0,1)\end{aligned}\tag{3.52}$$

Applying the inverse Sumudu transform to (3.49) and using (3.50) and (3.51),we get where

$$G_1^{\frac{3}{2}}(x,t)=\frac{1}{2\lambda}t^{-\frac{3}{4}}\phi\Big(-\frac{3}{4};\frac{1}{4};-\frac{|x|}{\lambda}t^{-\frac{3}{4}}\Big)$$

and

$$G_2^{\frac{3}{2}}(x,t)=\frac{1}{2\lambda}t^{\frac{1}{4}}\phi\Big(-\frac{3}{4};\frac{5}{4};-\frac{|x|}{\lambda}t^{-\frac{3}{4}}\Big)$$

4 Conclusion

In this paper, Sumudu transform of Caputo fractional derivatives have been used to solve fractional diffusion-wave equations. The solution of fractional diffusion-wave equations is obtained in terms of Mittag-Leffler function and generalized Wright function. The Sumudu transform and Fourier transform is an useful operational transform method which is an important in treating fractional diffusion-wave equations. The Sumudu transform and Fourier transform technique can be used to solve many types of initial value problems in applied and engineering fields.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Asiru MA. Application of the sumudu transform to discrete dynamical systems. International Journal of Mathematical Education, Science and Technology. 2003;34(6):944-949.

[2] Asiru MA. Further properties of the Sumudu transform and its applications. International journal of mathematical education in science and technology. 2002 May 1;33(3):441-9.

[3] Agarwal RP. A propos d'unc note de M.Pierre Humbert, C. R. Se'ances Acad. Sci. 1953 ;236(21): 2031-2032.

[4] Belgacem FB. Sumudu transform applications to Bessel functions and equations. Applied Mathematical Sciences. 2010;4(74):3665-86.

[5] Belgacem FB. Sumudu applications to Maxwell's equations. PIERS online. 2009;5(4):355-60.

[6] Belgacem FB. Introducing and analyzing deeper Sumudu properties. Nonlinear Studies. 2006 Feb 1;13(1):23-41.

[7] Dhaigude DB, Dhaigude CD. Linear initial value problems for fractional partial differential equations. Bull. Marathwada Math. Soc. 2012 Dec;13(2):17-38.

[8] Kataetbeh QD, Belgacem FBM. Applications of the Sumudu transform to differential equations. Nonlinear Studies. 2011;18(1):99-112.

[9] Watugala GK. Sumudu Transform for Functions of two Variables. Mathematical Engineering in Industry. 2002;8(4)293-302.

[10] Watugala G. Sumudu transform-a new integral transform to solve differential equations and control engineering problems. Mathematical engineering in industry. 1998 Jan 1;6(4):319-29.

[11] Watugala GK. Sumudu transform: a new integral transform to solve differential equations and control engineering problems. Integrated Education. 1993 Jan 1;24(1):35-43.

[12] Weerakoon S. Complex inversion formula for Sumudu transform. International Journal of Mathematical Education in Science and Technology. 1998;29(4):618-20.

[13] Weerakoon S. Applications of the Sumudu Transform to Partial Differential Equations. International Mathematical Education, Science and Technology. 1994;25(2):277-283.

[14] Belgacem FB, Karaballi AA, Kalla SL. Analytical investigations of the Sumudu transform and applications to integral production equations. Mathematical problems in Engineering. 2003 Dec 3;2003:103-18.

[15] Belgacem FBM, Karaballi AA. Sumudu transform fundamental properties investigations and applications. Journal of Applied Mathematics and Stochastic Analysis. 2006;1-23:Article ID 91083(2006).

[16] Lokenath D, Dambaru B. Integral transforms and their applications. Chapman and Hall /CRC, Taylor and Francis Group, New York; 2007.

[17] Miller KS, Ross B. An Introduction to the Fractional Calculus and Fractional Differential Equations. John Wiley and Sons; 1993.

[18] Karimov E, Ruzhansky M, Tokmagambetov N. Cauchy type problems for fractional differential equations. Integral Transforms and Special Functions. 2022 Jan 2;33(1):47-64.
DOI: 10.1080/10652469.2021.1900174.

[19] Belgacem R, Bokhari A, Sadaoui B. Shehu transform of Hilfer-Prabhakar fractional derivatives and applications on some Cauchy type problems. Advances in the Theory of Nonlinear Analysis and its Application. 2021 Jun 6;5(2):203-14.

[20] Bulut H, Baskonus HM, Belgacem FBM. The Analytical solution of some fractional ordinary differential equations by the Sumudu transform method, Abstract and Applied Analysis. 2013;6.

[21] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Application of Fractional Differential Equations. Elsevier, Amersterdam; 2006.

[22] Erdelyi(ed). A. Higher Transcendental Function, McGraw-Hill, New York. 1955; 3.

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