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On Some Recurrence Relations Connected with Generalized Fermat Numbers and Some Properties of Divisibility for these Numbers

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

As a result of nice properties of Fermat numbers and their interesting applications, these numbers have recently seen a variety of developments and extensions. Within this framework, this paper contributes. The purpose of this paper is to obtain some recurrence relations connected with generalized Fermat numbers $\mathcal{F}_n = a^{2^n} + 1$ for $a, n \in \mathbb{Z}$ and $n \geq 0$ and as a result of these recurrent relations, to get some properties of divisibility for generalized Fermat numbers.

Keywords: Fermat number; recurrence relation; divisibility of integers.

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1 INTRODUCTION

For $n \in \mathbb{Z}$ and $n \geq 0$, a Fermat number F_n is a positive integer of the form

$$F_n = 2^{2^n} + 1.$$

The first Fermat numbers are 3, 5, 17, 257, 65537, 4294967297, ... (sequence A000215 in the OEIS), see [1].

Since 1607–1665 when Fermat numbers were studied by the XVII-th century French lawyer and mathematician Pierre de Fermat, many people have been working about and around it, in many different directions and with a lot of applications in number theory, see [2], [3], [4], [5], [6]. The study of the various properties of the integers is often based on their arithmetic properties such as periodicity, congruences, and divisibility, all of which can be found in [7], [8] and [9].

Many mathematicians studied different aspects of the Fermat numbers during the XIX-th and XX-th centuries.

The Fermat number F_n for $n = 0, 1, 2, \ldots$ must always give a prime according to Fermat's conjecture in 1650. As a matter of fact, the first five Fermat numbers are prime numbers: $F_0 = 3, F_1 = 5$, $F_2 = 17, F_3 = 257, F_4 = 65537$. However, in 1732, L. Euler discovered that F_5 is a composite number:

$$F_5 = 2^{2^5} + 1 = 4294967297 = 641 \cdot 6700417,$$

where $641, 6700417 \in P$.

The set of Fermat factors starts with the numbers 3, 5, 17, 257, 641, 65537, 114689, 274177, 319489, 974849, ... (sequence $A023394$ in OEIS; see also sequence $A050922$ in OEIS), see [1].

As of 2020, 354 prime factors of Fermat numbers are known, and it is known that 310 Fermat numbers are composite. New Fermat factors are discovered each year. Subsequently, digital computers have facilitated the discovery of more factors. The distributed computing project Fermat Search is searching for new factors of Fermat numbers [10].

Calvo [11] reported some properties mainly related to the presence of common prime factors of both standard Fermat numbers, $F_n = 2^{2^n} + 1$, and generalized Fermat numbers, $\mathcal{F}_n = a^{2^n} + 1$.

Riesel [12] studied on a method to find certain common factors of numbers of the form $A_n = a^{2^n} + 1$.

Witno [13] investigated the behaviour of the numbers $E_n = \left(3^{2^n} + 1\right)/2$ and presented an initial primality test for E_n using the knowledge that E_{n-1} is prime.

In this paper, we aim to obtain some recurrent relations connected with the generalized Fermat numbers of the form $\mathcal{F}_n = a^{2^n} + 1$ for $a, n \in \mathbb{Z}$ and $n \geq 0$, and then to investigate some properties of divisibility of the generalized Fermat numbers.

2 RECURRENT RELATIONS FOR THE GENERALIZED FERMAT NUMBERS

In this section, we now give some new recurrent relations connected with the generalized Fermat numbers of the form $\mathcal{F}_n = a^{2^n} + 1$ for $a, n \in \mathbb{Z}$ and $n \geq 0$.

In the following proposition, we obtain the generalized Fermat number with index n, using the generalized Fermat numbers with the previous index $n-1$.

Proposition 2.1. *For $n \geq 1$, we have*

$$\mathcal{F}_n = (\mathcal{F}_{n-1} - 1)^2 + 1.$$

Proof. For $n \geq 1$, we have

$$\begin{aligned}\mathcal{F}_n - 1 &= a^{2^n} \\ &= a^{2^{n-1}.2} \\ &= \left(a^{2^{n-1}}\right)^2 \\ &= (\mathcal{F}_{n-1} - 1)^2.\end{aligned}$$

Thus, the proof is completed. □

In the following propositions, we obtain the generalized Fermat number with index n, using the generalized Fermat numbers with all previous indexes $0, 1, 2, \ldots, n-1$.

Proposition 2.2. *For $n \geq 1$, we have*

$$\mathcal{F}_n = (a-1)\,\mathcal{F}_0\mathcal{F}_1 \ldots \mathcal{F}_{n-1} + 2. \tag{2.1}$$

Proof. For $n \geq 1$, we have

$$\begin{aligned}
\mathcal{F}_n - 2 &= a^{2^n} - 1 \\
&= a^{2^{n-1}.2} - 1 \\
&= \left(a^{2^{n-1}}\right)^2 - 1 \\
&= \left(a^{2^{n-1}} + 1\right)\left(a^{2^{n-1}} - 1\right) \\
&= \left(a^{2^{n-1}} + 1\right)\left(a^{2^{n-2}} + 1\right)\left(a^{2^{n-2}} - 1\right) \\
&= \dots \\
&= \left(a^{2^{n-1}} + 1\right)\left(a^{2^{n-2}} + 1\right)\dots\left(a^{2^1} + 1\right)\left(a^{2^0} + 1\right)\left(a^{2^0} - 1\right) \\
&= \mathcal{F}_{n-1}\mathcal{F}_{n-2}\dots\mathcal{F}_0\,(a-1)\,.
\end{aligned}$$

Thus, the proof is completed. □

Proposition 2.3. *For $n \geq 2$, we have*

$$\mathcal{F}_n - \mathcal{F}_{n-1} = a^{2^{n-1}}\mathcal{F}_{n-2}\mathcal{F}_{n-3}\dots\mathcal{F}_0\,(a-1)\,.$$

Proof. For $n \geq 2$, we have

$$\begin{aligned}
\mathcal{F}_n - \mathcal{F}_{n-1} &= a^{2^n} - a^{2^{n-1}} \\
&= a^{2^{n-1}.2} - a^{2^{n-1}} \\
&= a^{2^{n-1}}a^{2^{n-1}} - a^{2^{n-1}} \\
&= a^{2^{n-1}}\left(a^{2^{n-1}} - 1\right) \\
&= a^{2^{n-1}}\left(a^{2^{n-2}.2} - 1\right) \\
&= a^{2^{n-1}}\left(\left(a^{2^{n-2}}\right)^2 - 1\right) \\
&= a^{2^{n-1}}\left(a^{2^{n-2}} + 1\right)\left(a^{2^{n-2}} - 1\right) \\
&= \dots \\
&= a^{2^{n-1}}\left(a^{2^{n-2}} + 1\right)\left(a^{2^{n-3}} + 1\right)\dots\left(a^{2^1} + 1\right)\left(a^{2^0} + 1\right)\left(a^{2^0} - 1\right) \\
&= a^{2^{n-1}}\mathcal{F}_{n-2}\mathcal{F}_{n-3}\dots\mathcal{F}_0\,(a-1)\,.
\end{aligned}$$

Thus, the proof is completed. □

Moreover, the generalized Fermat numbers satisfy the following recurrence relation.

Proposition 2.4. *For $n \geq 2$, we have*

$$\mathcal{F}_{n-1}^2 - 2\left(\mathcal{F}_{n-2} - 1\right)^2 = \mathcal{F}_n.$$

Proof. For $n \geq 2$, we have

$$\begin{aligned}
\mathcal{F}_{n-1}^2 - 2\left(\mathcal{F}_{n-2} - 1\right)^2 &= \left(a^{2^{n-1}} + 1\right)^2 - 2\cdot\left(a^{2^{n-2}}\right)^2 \\
&= \left(a^{2^{n-1}}\right)^2 + 1 + 2\cdot a^{2^{n-1}} - 2\cdot a^{2\cdot 2^{n-2}} \\
&= a^{2^n} + 1 \\
&= \mathcal{F}_n.
\end{aligned}$$

Thus, the proof is completed. □

3 DIVISIBILITY PROPERTIES OF THE GENERALIZED FERMAT NUMBERS

Using the results reached before, we obtain some divisibility properties of the generalized Fermat numbers $\mathcal{F}_n$ in this section.

Theorem 3.1. *For $m \neq n$, we have $gcd\left(\mathcal{F}_m, \mathcal{F}_n\right) = 1$ or $gcd\left(\mathcal{F}_m, \mathcal{F}_n\right) = 2$.*

Proof. In order to prove $gcd\left(\mathcal{F}_m, \mathcal{F}_n\right) = 1$ or $gcd\left(\mathcal{F}_m, \mathcal{F}_n\right) = 2$, let assume that $gcd\left(\mathcal{F}_m, \mathcal{F}_n\right) = d$. In this case, it holds that $d \mid \mathcal{F}_n$ and $d \mid \mathcal{F}_m$. Without loss of generality, let $n > m$. Since $d \mid \mathcal{F}_n$, from (2.1) we get

$$d \mid (a-1)\,\mathcal{F}_0\mathcal{F}_1 \ldots \mathcal{F}_m \ldots \mathcal{F}_{n-1} + 2. \qquad (3.1)$$

Also, as $d \mid \mathcal{F}_m$, then

$$d \mid (a-1)\,\mathcal{F}_0\mathcal{F}_1 \ldots \mathcal{F}_m \ldots \mathcal{F}_{n-1}. \qquad (3.2)$$

Moreover, if $k = (a-1)\,\mathcal{F}_0\mathcal{F}_1 \ldots \mathcal{F}_m \ldots \mathcal{F}_{n-1}$, then using relations in (3.1) and (3.2), it is easy to see that

$$d \mid [(k+2) - k] = 2.$$

Thus, it is proven that $gcd\left(\mathcal{F}_m, \mathcal{F}_n\right) = 1$ or $gcd\left(\mathcal{F}_m, \mathcal{F}_n\right) = 2$. □

Corollary 3.2. *Let a in the nth generalized Fermat number $\mathcal{F}_n = a^{2^n} + 1$ be an even integer. Then, the numbers $\mathcal{F}_m$ and $\mathcal{F}_n$ have no common prime divisors, i.e., for $m \neq n$, we have $gcd\left(\mathcal{F}_m, \mathcal{F}_n\right) = 1$.*

Proof. Let a in the nth generalized Fermat number $\mathcal{F}_n = a^{2^n} + 1$ be an even integer. In order to prove $gcd\left(\mathcal{F}_m, \mathcal{F}_n\right) = 1$ for $m \neq n$, let assume that $gcd\left(\mathcal{F}_m, \mathcal{F}_n\right) = d$ and $d > 1$. Since every integer $n \geq 2$ has a prime factor, the integer $d \geq 2$ has a prime factor. Let p be a prime factor of d. In this case, it holds that $p \mid d$. On the other hand, since $gcd\left(\mathcal{F}_m, \mathcal{F}_n\right) = d$ and $p \mid d$, it is easy to see that $p \mid F_n$, and $p \mid F_m$. Moreover, if $k = (a-1)\,\mathcal{F}_0\mathcal{F}_1 \ldots \mathcal{F}_m \ldots \mathcal{F}_{n-1}$, then using relations in (3.1) and (3.2), we have that

$$p \mid [(k+2) - k] = 2.$$

Therefore, since p is a prime factor, we obtain that $p = 2$. Also, since a is an even integer, any nth generalized Fermat number $\mathcal{F}_n = a^{2^n} + 1$ is an odd integer. Thus, it is not divisible by 2. It gives a contradiction. Consequently, we obtain $gcd\left(\mathcal{F}_m, \mathcal{F}_n\right) = 1$ for $n \neq m$. □

This fact allow as to give one more proof of infiniteness of the set of prime numbers.

Theorem 3.3. *There are infinity many prime numbers.*

Proof. Let a in the nth generalized Fermat number $\mathcal{F}_n = a^{2^n} + 1$ be an even positive integer. For $n = 0, 1, 2, ...$, the sequence $\{\mathcal{F}_n\}_{n \geq 0}$ is an increasing infinite sequence such that

$$\{\mathcal{F}_n\}_{n \geq 0} = \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n, \ldots\}.$$

If we now denote by p_n the smallest prime divisor of each F_n in $\{\mathcal{F}_n\}_{n \geq 0}$, then from Corollary 3.2 we get $p_n \neq p_m$ for $n \neq m$. Therefore, all elements of the set $P = \{p_0, p_1, p_2, \ldots, p_n, \ldots\}$ are different odd prime numbers and the set P is an infinite subset of the set of primes. Thus, we obtain that the set of all primes is infinite. □

4 CONCLUSIONS

In this paper, about the generalized Fermat numbers of the form $\mathcal{F}_n = a^{2^n} + 1$ for $a, n \in \mathbb{Z}$ and $n \geq 0$, we first obtained some new recurrent relations. Then, we established some divisibility properties for the generalized Fermat numbers $\mathcal{F}_n$. Furthermore, we demonstrate that there are infinitely many prime numbers from a different viewpoint.

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COMPETING INTERESTS

Author has declared that no competing interests exist.

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