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A New Iterative Algorithm for Total Asymptotically Non-Expansive Mapping in CAT(0) Spaces

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, we provide certain fixed point results for a total asymptotically non-expansive mapping, as well as a new iterative algorithm for approximating the fixed point of this class of mappings in the setting of CAT(0) spaces. Furthermore, we establish strong and Δ-converges theorem for total asymptotically non-expansive mapping in CAT(0) space. Our result, generalizes, improve, extend and unify the results of Thakur et al. [1], Izhar et al. [2] and many more in this direction.

Keywords: CAT (0) space; total asymptotically non-expansive mappings; strong and Δ-convergence theorems.

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1 Introduction

Once the existence of a solution for an operator equation is confirmed, such a solution cannot be derived analytically in many circumstances. To overcome such situations, one must approximate the value of the answer. To accomplish this, we first arrange the operator equation in the form of a fixed point equation, then apply the most appropriate iterative algorithm to the fixed point equation, then the limit of the sequence generated by the most appropriate iterative algorithm. In actuality, the desired fixed point value for the fixed point equation and the operator equation solution. In this direction in 1920, Banach fixed point theorem enforced and suggest a simple iterative algorithm $\breve{x}_{m+1} = S\breve{x}_m$ (Picard iterative algorithm) for contraction mapping, but it fails to converge the fixed point for non-expansive mappings, to illustrated, $S\colon [0,1] \to [0,1]$ defined by $S\breve{x} = 1 - \breve{x}$, with initial choice $\breve{x}_0 \neq 1/2$, Picard iterative algorithm is not convergent. Then natural question arise:

Question: Does there exist an iterative algorithm which converges for non-expansive mapping in suitable spaces ? The affirmative answer was gives by Mann[3], Ishikawa[4], Noor[5] not only for non-expansive mapping but also for more general then non-expansive mapping in various topological space.

Naturally the various researcher attract in this direction, if the iterative algorithm exists for approximation fixed point, which converges to faster, toward fixed point. Affirmative approach was gives by Agarwal et al.[6] in 2007, by introduce S-iteration process defined as follows: Let Θ be a convex subset of normed space X and a non-linear mapping S of Θ into itself, the sequence $\{\varsigma_m\}$ in Θ defined by

$$\varsigma_1 \in \Theta$$
$$\varsigma_{m+1} = (1-\alpha_m)S\varsigma_m + \alpha_m S\nu_m$$
$$\nu_m = (1-\beta_m)\varsigma_m + \beta_m S\varsigma_m \quad \forall m \geq 1,$$

where $\{\varsigma_m\}$ is real sequence in (0,1). After the development of S- iteration process,numerous researchers developed iteration processes, including Thakur et al. [1], modified Picard S-hybrid iteration [7], modified Picard Mann hybrid [1]. In the last decades, so many researchers attract to approximate fixed point in CAT(0) space (see [8],[9]). Now we discuss about CAT(0) space and there properties. A CAT(0) space is one in which all geodesic triangles of acceptable size satisfy the following CAT(0) comparison axiom. Let Δ be a geodesic triangle in Θ and let $\bar{\Delta} \subset \mathbb{R}^2$ be comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if for all $\breve{x}, \breve{y} \in \Delta$ and all comparison points $\bar{\breve{x}}, \bar{\breve{y}} \in \bar{\Delta}$,

$$\wp(\breve{x}, \breve{y}) \leq \wp_{\mathbb{R}^2}(\bar{\breve{x}}, \bar{\breve{y}}).$$

If $\breve{x}, \breve{y}_1, \breve{y}_2$ are points of a CAT(0) space and y_0 is the midpoint of the segment $[\breve{y}_1, \breve{y}_2]$ which we will denote by $(\breve{y}_1 \oplus \breve{y}_2)/2$, then the CAT(0) inequality implies

$$\wp^2(\breve{x}, \frac{\breve{y}_1 \oplus \breve{y}_2}{2}) \leq \frac{1}{2}\wp^2(\breve{x}, \breve{y}_1) + \frac{1}{2}\wp^2(\breve{x}, \breve{y}_2) - \frac{1}{4}\wp^2(\breve{y}_1, \breve{y}_2).$$

this inequality is the (CN) inequality of Bruhat and Tits [10]. It is well known that all complete, simply combined Riemannian manifold having non-positive section curvature is a CAT(0) space. For other examples, Euclidean buildings [11], Pre-Hilbert spaces, $\mathbb{R}$-trees [12], the complex Hilbert ball with a hyperbolic metric [9] is a CAT(0) space. Further, complete CAT(0) spaces are called Hadamard spaces. In the sequel, we give some fundamental for nonlinear mappings in CAT(0) spaces. Let Θ be a nonempty closed subset of a CAT(0) space, M and S be a self map defined on Θ. Then S is said to be:

1. non-expansive if
$$\wp(S^m\varsigma, S^m\nu) \leq \wp(\varsigma, \nu), \forall \varsigma, \nu \in \Theta;$$
2. asymptotically non-expansive if $\exists$ a sequence $\{\xi_m\}$ in $[1, \infty)$ with $lim_{m\to\infty}\xi_m = 1$ such that
$$\wp(S^m\varsigma, S^m\nu) \leq \xi_m\wp(\varsigma, \nu), \forall \varsigma, \nu \in \Theta \ and \ \forall m \geq 1;$$

3. uniformly L-Lipschitzian if $\exists$ a constant $L > 0$ such that

$$\wp(S^m\varsigma, S^m\nu) \leq L\wp(\varsigma, \nu), \forall \varsigma, \nu \in \Theta \ and \ \forall m \geq 1;$$

Goebel and Kirk [9] presented the class of asymptotically non-expansive mappings as a generali-zation of the class of non-expansive maps in 1972. They established that if Θ is a nonempty closed convex bounded subset of a real uniformly convex Banach space and S is an asymptotically nonexpan- sive self-mapping of Θ, S has a fixed point. Following that, numerous researchers were inclined to phrase the approximating the fixed point for asymptotically non-expansive mapping via, Mann [3], Ishikawa [4] and Noor [5] in various topological spaces (see in [13],[14],[15],[16]). Bruck et al. [17], proposed asymptotically non-expansive intermediate sense mappings.

Definition 1.1. If an operator s is continuous and the following condition holds, it is said to be asymptotically non-expansive in the intermediate sense:

$$\lim_{m\to\infty} \sup \sup_{\varsigma,\nu\in\Theta} (\wp(S^m\varsigma, S^m\nu) - \wp(\varsigma, \nu)). \tag{1.1}$$

Observe that if

$$\breve{a}_n :\ = \sup_{\varsigma,\nu\in\Theta} (\wp(S^m\varsigma, S^m\nu) - \wp(\varsigma, \nu)). \tag{1.2}$$

then (1.1) reduces to the relation

$$\wp(S^m\varsigma, S^m\nu) \leq \wp(\varsigma, \nu) + \breve{a}_n \qquad \forall \varsigma, \nu \in \Theta. \tag{1.3}$$

It is worth noting that the class of asymptotically non-expansive maps is correctly included within the class of asymptotically non-expansive mappings in the intermediate sense (see, e.g., [18]). Alber et al. [19] initially established the concept of total asymptotically non-expansive mappings in 2006 (see also [20],[21],[18],[7]) defined as follows.

Definition 1.2. A self mapping S on Θ is called $(\{\mu_m\}, \{\varkappa_m\}, \varphi)$ total asymptotically non-expansive mapping if there exist nonnegative real sequences $\{\mu_m,\}$ and $\{\varkappa_m\}$ with $\mu_m \to 0$, $\varkappa_m \to 0$ as $m \to \infty$ and a continuous strictly increasing function $\varphi \colon [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that

$$\wp(S^m\varsigma, S^m\nu) \leq \wp(\varsigma, \nu) + \mu_m\varphi(\wp(\varsigma, \nu)) + \varkappa_m. \tag{1.4}$$

$\forall \varsigma, \nu \in \Theta \ and \ \forall m \geq 1.$

Remark. If $\phi(\lambda) = \lambda$, then (1.4) takes the form

$$\wp(S^m\varsigma, S^m\nu) \leq (1 + \mu_m)\wp(\varsigma, \nu) + \varkappa_m. \tag{1.5}$$

Remark. In addition, if $\varkappa_m = 0$ for all $m \geq 1$, then total asymptotically non-expansive mappings coincide with asymptotically non-expansive mappings.

Remark. If $\mu_m = 0$ and $\varkappa_m = 0$ for all $m \geq 1$, then we obtain from (1.4) the class of non-expansive mappings.

In 2012, Chang et al. [22] demiclosedness principle for total asymptotically non-expansive mapping in CAT(0) space and established strong and $Delta$-convergence theorem for Mann iteration process. As a result, other researchers established strong and $Delta$-convergence theorem for the same mapping through modified Ishikawa [23], S-iteration process [24], modified Picard S-hybrid iteration [7], modified Picard Mann hybrid [1] in CAT(0) spaces. Recently, Izhar et al. [2] presented A modified iteration process $\{\varsigma_m\}$ and $\{\nu_m\}$, which is defined below:

$$\varsigma_1 \in \Theta$$

$$\omega_m = S^m((1 - \alpha_m)\varsigma_m \oplus \alpha_m S^m\varsigma_m)$$
$$\nu_m = S^m((1 - \beta_m)S^m\omega_m \oplus \beta_m S^m\omega_m)$$

$$\varsigma_{m+1} = S^m \nu_m \tag{1.6}$$

for all $m \geq 1$, where $\{\alpha_m\}$ and $\{\beta_m\}$ are appropriate sequences in the interval (0,1) and They established various convergence theorems of the iterative scheme created sequence (1.6) to approxim- ate the fixed point of total asymptotically non-expansive mapping in CAT(0) spaces.

Motivated by above work, we introduce a new iteration process in CAT(0) space. Let Θ be a nonempty subset of CAT(0) space. $S\colon \Theta \to \Theta$ be a total asymptotically non-expansive mapping. Let $\{e_m\}$ be sequence in Θ define iteratively as follows:

$$\begin{aligned} e_1 &\in \Theta \\ e_{m+1} &= S^m f_m \\ f_m &= S^m[(1-\alpha_m)g_m \oplus \alpha_m S^m g_m] \\ g_m &= S^m[(1-\beta_m)h_m \oplus \beta_m S^m h_m] \\ h_m &= S^m[(1-\gamma_m)e_m \oplus \gamma_m S^m e_m] \end{aligned} \tag{1.7}$$

for all $m \geq 1$, where $\{\alpha_m\}$,$\{\beta_m\}$ and $\{\gamma_m\}$ are appropriate sequences in the interval (0,1). In this paper we prove various iterative sequence convergence theorems (1.7) for total asymptotically non-expansive mapping in CAT(0) spaces and we also show numerical examples to compare the rate of convergence of our iteration process with several existing iteration processes.Our result, generalize the result of Izhar et al.[2] and many more in this direction.

2 Preliminaries

This section contains some well-known concepts and results that will be referenced throughout the paper.

Lemma 2.1. ([8]) Let M be a CAT(0) space, $\breve{a}, \breve{b}, \breve{c} \in M$ and $t \in [0,1]$. Then

$$\wp(t\breve{a} \oplus (1-t)\breve{b}, \breve{c}) \leq t\wp(\breve{a}, \breve{c}) + (1-t)\wp(\breve{b}, \breve{c}).$$

Let $\{\breve{a}_m\}$ be a bounded sequence in M, complete CAT(0) spaces. For $\breve{a} \in M$ set:

$$r(\breve{a}, \{\breve{a}_m\}) = \lim_{m\to\infty} sup\ \wp(\breve{a}, \breve{a}_m).$$

The asymptotic radius $r(\{\breve{a}_m\})$ is given by

$$r(\{\breve{a}_m\}) = inf\{r(\breve{a}, \breve{a}_m)\colon \breve{a} \in M\}.$$

and the asymptotic center $\mathcal{A}(\{\breve{a}_m\})$ of $\{\breve{a}_m\}$ is defined as:

$$\mathcal{A}(\{\breve{a}_m\}) = \{\breve{a} \in M\colon r(\breve{a}, \breve{a}_m) = r(\{\breve{a}_m\})\}.$$

$\mathcal{A}(\{\breve{a}_m\})$ consists of exactly one point in CAT(0) spaces see ([25], Proposition 7). A sequence $\{\breve{a}_m\}$ in M is said to Δ-converges to $\breve{a} \in M$ if a is the unique asymptotic center for every subsequence $\{\breve{z}_m\}$ of $\{\breve{a}_m\}$. In this case we write $\Delta - lim_n \breve{a}_m = \breve{a}$ and read as $\breve{a}$ is the $\Delta - limit$ of $\{\breve{a}_m\}$.

Lemma 2.2. ([26]) Let M be a complete CAT(0) space and $\{\breve{a}_m\}$ be a bounded sequence in M. If $\mathcal{A}(\{\breve{a}_m\}) = \{\rho\}, \{\breve{z}_m\}$ is a subsequence of $\{\breve{a}_m\}$ such that $\mathcal{A}(\{\breve{z}_m\})) = \{\breve{z}\}$ and $\wp(\breve{a}_m, \breve{z})$ converges, then $\rho = \breve{z}$.

Recall that according to Karapinar et al. [27], the fixed point existence theorem and the demiclose- dness principle for mappings fulfil Definition 1.2 in CAT(0) spaces.

Lemma 2.3. ([27]) Suppose $S\colon \Theta \to \Theta$ be a nonempty convex closed and bounded subset of M, a complete CAT(0) space. Let S be uniformly continuous and total asymptotically non-expansive mapping. Then, S has a fixed point and set of fixed points $\mathcal{F}(S)$ is convex and closed.

Lemma 2.4. ([27]) Suppose $S\colon \Theta \to \Theta$ be a nonempty closed, convex subset of M, a complete CAT(0) space. Let S be a uniformly continuous and total asymptotically non-expansive mapping. For every bounded sequence $\{\breve{z}_m\} \in \Theta$ such that, $\lim_{m\to\infty} \wp(\breve{a}_m, S\breve{a}_m) = 0$ and $\lim_{m\to\infty} \breve{a}_m = q$ implies that $Sq = q$.

Lemma 2.5. ([28]) Let M be a complete CAT(0) space and let $\breve{a} \in M$. Suppose $\{t_m\}$ is a sequence in $[\breve{b}, e]$ for some $\breve{b}, e \in (0,1)$ and $\{\breve{a}_m\}$, $\{\breve{b}_m\}$ are sequences in M such that $\lim_{m\to\infty} sup\ \wp(\breve{a}_m, \breve{a}) \le r$, $\lim_{m\to\infty} sup\ \wp(\breve{b}_m, \breve{a}) \le r$ and $\lim_{m\to\infty} \wp((1-t_m)\breve{a}_m \oplus t_m\breve{b}_m, \breve{a}) = r$ for some $r \ge 0$. Then

$$\lim_{m\to\infty} \wp(\breve{a}_m, \breve{b}_m) = 0.$$

Lemma 2.6. ([29]) Let $\{\alpha_m\}$, $\{\beta_m\}$ and $\{\zeta_m\}$ be the sequences of nonnegative numbers such that

$$\alpha_{m+1} \le (1+\beta_m)\alpha_m + \zeta_m.$$

For all $m \ge 1$. $\sum_{m=1}^{\infty} \beta_m < \infty$ and $\sum_{m=1}^{\infty} \zeta_m < \infty$, then $\lim_{m\to\infty} \alpha_m$ exists. Whenever, if $\exists$ a subsequence $\{\alpha_{m_k}\} \subseteq \{\alpha_m\}$ such that $\alpha_{m_k} \to 0$ as $k \to \infty$, then $\lim_{m\to\infty} \alpha_m = 0$.

3 Main Result

Theorem 3.1. *Let Θ be a closed bounded and convex subset of M, a complete CAT(0) space and $S\colon \Theta \to \Theta$ is uniformly L-Lipschitzian and $(\{\mu_m\}, \{\varkappa_m\}, \varphi)$-total asymptotically non-expansive mapping. Assume that the following conditions hold:*
(a) $\sum_{m=1}^{\infty} \mu_m < \infty$ and $\sum_{m=1}^{\infty} \varkappa_m < \infty$;
(b) $\exists$ constants $\breve{m}, \breve{n}$ with $0 < \breve{m} \le \gamma_m \le \breve{n} < 1$ $\forall m \in N$.
(c) $\exists$ constants $\breve{p}, \breve{q}$ with $0 < \breve{p} \le \beta_m \le \breve{q} < 1$ $\forall m \in N$.
(d) $\exists$ constants $\breve{l}, \breve{o}$ with $0 < \breve{l} \le \alpha_m \le \breve{o} < 1$ $\forall m \in N$.
(e) $\exists$ constants M^* such that $\varphi(\breve{w}) \le M^*\breve{w}$ $\forall \breve{w} \ge 0$.
Then the sequence $\{e_m\}$ defined by Δ-converges to a point of $F(S) = \{\breve{x} \in \Theta\colon S(\breve{x}) = \breve{x}\}$ is the set of fixed point.

Proof. By using Lemma 2.5, we have $F(S) \neq \emptyset$. We begin by demonstrating that $\lim_{m\to\infty} \wp(e_m, \rho)$ exists for any $\rho \in F(S)$, where $\{e_m\}$ is defined by (1.7). Let $\rho \in F(S)$. Then we have

$$\begin{aligned}
\wp(h_m, \rho) &= \wp(S^m((1-\gamma_m)e_m \oplus \gamma_m S^m e_m), \rho). \\
&\le \wp((1-\gamma_m)e_m \oplus \gamma_m S^m e_m), \rho) \\
&\quad + \mu_m \varphi\wp((1-\gamma_m)e_m \oplus \gamma_m S^m e_m), \rho) + \varkappa_m. \\
&\le (1+\mu_m M^*)\wp((1-\gamma_m)e_m \oplus \gamma_m S^m e_m), \rho) + \varkappa_m. \\
&\le (1+\mu_m M^*)[(1-\gamma_m)\wp(e_m, \rho) + \gamma_m S^m \wp(e_m, \rho)] + \varkappa_m. \\
&\le (1+\mu_m M^*)[(1-\gamma_m)\wp(e_m, \rho) + \mu_m \varphi\wp(e_m, \rho)) \\
&\quad + \varkappa_m] + \varkappa_m. \\
&\le (1+\mu_m M^*)[(1+\mu_m M^*)\wp(e_m, \rho)) + \varkappa_m] + \varkappa_m. \\
&\le (1+\mu_m M^*)^2 \wp(e_m, \rho) + (2+\mu_m M^*)\varkappa_m.
\end{aligned} \tag{3.1}$$

$\forall m \in N$ using (3.1). We have

$$\begin{aligned}
\wp(g_m,\rho) &= \wp(S^m((1-\beta_m)h_m \oplus \beta_m S^m h_m),\rho).\\
&\le \wp((1-\beta_m)h_m \oplus \beta_m S^m h_m),\rho)\\
&\quad + \mu_m\varphi\wp((1-\beta_m)h_m \oplus \beta_m S^m h_m),\rho) + \varkappa_m.\\
&\le (1+\mu_m M^*)\wp((1-\beta_m)h_m \oplus \beta_m S^m h_m),\rho) + \varkappa_m.\\
&\le (1+\mu_m M^*)[(1-\beta_m)\wp(h_m,\rho) + \mu_m\varphi\wp(h_m,\rho))\\
&\quad + \varkappa_m] + \varkappa_m.\\
&\le (1+\mu_m M^*)[(1+\mu_m M^*)\wp(h_m,\rho)) + \varkappa_m] + \varkappa_m.\\
&\le (1+\mu_m M^*)^2\wp(h_m,\rho) + (2+\mu_m M^*)\varkappa_m.\\
&\le (1+\mu_m M^*)^2[(1+\mu_m M^*)^2\wp(e_m,\rho) + (2+\mu_m M^*)\varkappa_m]\\
&\quad + (2+\mu_m M^*)\varkappa_m.\\
&\le (1+\mu_m M^*)^4\wp(e_m,\rho) + [1+(1+\mu_m M^*)^2](2+\mu_m M^*)\varkappa_m.
\end{aligned} \tag{3.2}$$

$\forall m \in N$ using (3.2). Also we have

$$\begin{aligned}
\wp(f_m,\rho) &= \wp(S^m((1-\alpha_m)g_m \oplus \alpha_m S^m g_m),\rho).\\
&\le \wp((1-\alpha_m)g_m \oplus \alpha_m S^m g_m),\rho)\\
&\quad + \mu_m\varphi\wp((1-\alpha_m)g_m \oplus \alpha_m S^m g_m),\rho) + \varkappa_m.\\
&\le (1+\mu_m M^*)\wp((1-\alpha_m)g_m \oplus \alpha_m S^m g_m),\rho) + \varkappa_m.\\
&\le (1+\mu_m M^*)[(1-\alpha_m)\wp(g_m,\rho) + \mu_m\varphi\wp(g_m,\rho))\\
&\quad + \varkappa_m] + \varkappa_m.\\
&\le (1+\mu_m M^*)[(1+\mu_m M^*)\wp(g_m,\rho)) + \varkappa_m] + \varkappa_m.\\
&\le (1+\mu_m M^*)^2\wp(g_m,\rho) + (2+\mu_m M^*)\varkappa_m.\\
&\le (1+\mu_m M^*)^2[(1+\mu_m M^*)^4\wp(e_m,\rho)\\
&\quad + [1+(1+\mu_m M^*)^2](2+\mu_m M^*)\varkappa_m] + (2+\mu_m M^*)\varkappa_m.\\
&\le (1+\mu_m M^*)^6\wp(e_m,\rho)\\
&\quad + (1+\mu_m M^*)^2[1+(1+(1+\mu_m M^*)^2)](2+\mu_m M^*)\varkappa_m.
\end{aligned} \tag{3.3}$$

$\forall m \in N$. From(1.7), (3.1), (3.2) and (3.3), we get

$$\begin{aligned}
\wp(e_{m+1},\rho) &= \wp(S^m f_m,\rho).\\
&\le \wp(f_m,\rho) + \mu_m\varphi\wp(f_m,\rho) + \varkappa_m.\\
&\le (1+\mu_m M^*)\wp(f_m,\rho) + \varkappa_m.\\
&\le (1+\mu_m M^*)[(1+\mu_m M^*)^6\wp(e_m,\rho)\\
&\quad + (1+\mu_m M^*)^2(1+(1+((1+\mu_m M^*)^2))(2+\mu_m M^*)\varkappa_m] + \varkappa_m.\\
&\le (1+\mu_m M^*)^7\wp(e_m,\rho)\\
&\quad + [1+(1+\mu_m M^*)^3(1+(1+((1+\mu_m M^*)^2))(2+\mu_m M^*)]\varkappa_m.
\end{aligned} \tag{3.4}$$

Where

$$\eta_m\colon\ = (1+\mu_m M^*)^7 \quad and \quad \xi_m\colon\ = 1+(1+\mu_m M^*)^3(2+\mu_m M^*)(1+(1+((1+\mu_m M^*)^2)).$$

By assumption (a), we have

$$\sum_{m=1}^{\infty}\eta_m < \infty \quad and \quad \sum_{m=1}^{\infty}\xi_m < \infty. \tag{3.5}$$

By assertion (3.4), (3.5) and Lemma 2.6, we obtain $\lim_{m\to\infty} \wp(e_m, \rho)$ exists.

Next, we prove that $\lim_{m\to\infty} \wp(e_m, S^m e_m) = 0$.

Without loss of generality we may suppose that

$$\lim_{m\to\infty} \wp(e_m, \rho) = \breve{w} \geq 0. \tag{3.6}$$

From (3.1),we have

$$\lim_{m\to\infty} sup\ \wp(h_m, \rho) \leq \breve{w}. \tag{3.7}$$

Since S satisfies Definition 1.2

$$\wp(S^m h_m, \rho) \leq \wp(h_m, \rho) + \mu_m \varphi \wp(h_m, \rho) + \varkappa_m.$$

$$\wp(S^m h_m, \rho) \leq (1 + \mu_m M^*)\wp(h_m, \rho) + \varkappa_m. \tag{3.8}$$

From (3.7) and (3.8), we have

$$\lim_{m\to\infty} sup\ \wp(S^m h_m, \rho) \leq \breve{w}. \tag{3.9}$$

In the same way, we get

$$\lim_{m\to\infty} sup\ \wp(S^m e_m, \rho) \leq \breve{w}. \tag{3.10}$$

Since

$$\begin{aligned}\wp(e_{m+1}, \rho) &\leq (1 + \mu_m M^*)^7 \wp(e_m, \rho) \\ &\quad + [1 + (1 + \mu_m M^*)^3 (1 + (1 + ((1 + \mu_m M^*)^2))(2 + \mu_m M^*)]\varkappa_m.\end{aligned}$$

By taking limit infimum both sides, we obtain,

$$\breve{w} \leq \lim_{m\to\infty} inf\ \wp(h_m, \rho). \tag{3.11}$$

By using (3.7) and (3.11), we have

$$\breve{w} = \lim_{m\to\infty} sup\ \wp(h_m, \rho) \leq \lim_{m\to\infty} sup\ \wp(S^m((1 - \gamma_m)e_m \oplus \gamma_m S^m e_m), \rho). \tag{3.12}$$

$$\begin{aligned}\wp(S^m((1 - \gamma_m)e_m \oplus \gamma_m S^m e_m), \rho) &\leq \wp((1 - \gamma_m)e_m \oplus \gamma_m S^m e_m), \rho) \\ &\quad + \mu_m \varphi \wp((1 - \gamma_m)e_m \oplus \gamma_m S^m e_m), \rho) + \varkappa_m. \\ &\leq (1 + \mu_m M^*)\wp((1 - \gamma_m)e_m \oplus \gamma_m S^m e_m), \rho) + \varkappa_m.\end{aligned}$$

$$\lim_{m\to\infty} sup\ \wp(S^m((1 - \gamma_m)e_m \oplus \gamma_m S^m e_m), \rho) \leq \lim_{m\to\infty} sup\ \wp((1 - \gamma_m)e_m \oplus \gamma_m S^m e_m), \rho).$$

$$\breve{w} \leq \lim_{m\to\infty} sup\ \wp((1 - \gamma_m)e_m \oplus \gamma_m S^m e_m), \rho). \tag{3.13}$$

By using (3.6) and (3.10), we have

$$\wp((1 - \gamma_m)e_m \oplus \gamma_m S^m e_m), \rho) \leq ((1 - \gamma_m)\wp(e_m, \rho) + \gamma_m \wp(S^m e_m, \rho).$$

By taking limit suprimum both sides, we obtain

$$\lim_{m\to\infty} sup\ \wp((1 - \gamma_m)e_m \oplus \gamma_m S^m e_m), \rho) \leq \breve{w}. \tag{3.14}$$

Applying (3.13) and (3.14), we have,

$$\lim_{m\to\infty} sup\ \wp((1 - \gamma_m)e_m \oplus \gamma_m S^m e_m), \rho) = \breve{w}. \tag{3.15}$$

By using (3.6), (3.10), (3.15) and Lemma 2.5, we can conclude that

$$\lim_{m\to\infty} \wp(e_m, S^m e_m) = 0. \tag{3.16}$$

We also have,

$$\wp(e_{m+1},\rho) \leq (1+\mu_m M^*)\wp(f_m,\rho) + \varkappa_m.$$
$$\wp(e_{m+1},\rho) \leq (1+\mu_m M^*)[(1+\mu_m M^*)^2\wp(g_m,\rho) + (2+\mu_m M^*)\varkappa_m] + \varkappa_m.$$
$$\wp(e_{m+1},\rho) \leq (1+\mu_m M^*)^3\wp(g_m,\rho) + [1+(1+\mu_m M^*)(2+\mu_m M^*)]\varkappa_m.$$

By taking limit infimum both sides, we obtain

$$\breve{w} \leq \lim_{m\to\infty} inf\ \wp(g_m,\rho). \tag{3.17}$$

$$\wp(g_m,\rho) \leq (1+\mu_m M^*)^4\wp(e_m,\rho) + [1+(1+\mu_m M^*)^2](2+\mu_m M^*)\varkappa_m.$$

By taking limit suprimum both sides, we obtain

$$\lim_{m\to\infty} sup\ \wp(g_m,\rho) \leq \breve{w}. \tag{3.18}$$

By using (3.17) and (3.18), we get

$$\breve{w} = \lim_{m\to\infty} sup\ \wp(g_m,\rho) = \lim_{m\to\infty} sup\ \wp(S^m((1-\beta_m)h_m \oplus \beta_m S^m h_m),\rho). \tag{3.19}$$

$$\begin{aligned}\wp(S^m((1-\beta_m)h_m \oplus \beta_m S^m h_m),\rho) &\leq \wp((1-\beta_m)h_m \oplus \beta_m S^m h_m,\rho)\\ &\quad + \mu_m\varphi\wp((1-\beta_m)h_m \oplus \beta_m S^m h_m,\rho) + \varkappa_m.\\ &\leq (1+\mu_m M^*)\wp((1-\beta_m)h_m \oplus \beta_m S^m h_m,\rho) + \varkappa_m.\end{aligned}$$

$$\lim_{m\to\infty} sup\ \wp(S^m((1-\beta_m)h_m \oplus \beta_m S^m h_m),\rho) \leq \lim_{m\to\infty} sup\ \wp((1-\beta_m)h_m \oplus \beta_m S^m h_m,\rho).$$

$$\breve{w} \leq \lim_{m\to\infty} inf\ \wp((1-\beta_m)h_m \oplus \beta_m S^m h_m,\rho). \tag{3.20}$$

Also we have

$$\wp((1-\beta_m)h_m \oplus \beta_m S^m h_m,\rho) \leq (1-\beta_m)\wp(h_m,\rho) + \beta_m\wp(S^m h_m,\rho).$$

By taking limit suprimum both sides, we obtain

$$\lim_{m\to\infty} sup\ \wp((1-\beta_m)h_m \oplus \beta_m S^m h_m,\rho) \leq \breve{w}. \tag{3.21}$$

By using (3.9), (3.12), (3.21) and Lemma 2.5, we can conclude that

$$\lim_{m\to\infty} \wp(h_m, S^m h_m) = 0. \tag{3.22}$$

Since S is $(\{\mu_m\},\{\varkappa_m\},\varphi)$-total asymptotically non-expansive mapping.

$$\begin{aligned}\wp(S^m h_m, S^m e_m) &\leq \wp(h_m,e_m) + \mu_m\varphi\wp(h_m,e_m) + \varkappa_m.\\ &\leq (1+\mu_m M^*)\wp(h_m,e_m) + \varkappa_m.\\ &\leq (1+\mu_m M^*)\wp(S^m((1-\gamma_m)e_m \oplus \gamma_m S^m e_m),e_m) + \varkappa_m.\\ &\leq (1+\mu_m M^*)\wp(S^m((1-\gamma_m)e_m \oplus \gamma_m S^m e_m),S^m e_m)\\ &\quad + (1+\mu_m M^*)\wp(S^m e_m,e_m) + \varkappa_m.\\ &\leq (1+\mu_m M^*)[\wp((1-\gamma_m)e_m \oplus \gamma_m S^m e_m,e_m)\\ &\quad + \mu_m M^*\wp((1-\gamma_m)e_m \oplus \gamma_m S^m e_m,e_m) + \varkappa_m]\\ &\quad + (1+\mu_m M^*)\wp(S^m e_m,e_m) + \varkappa_m.\\ &\leq (1+\mu_m M^*)^2[\gamma_m\wp(S^m e_m,e_m)] + (1+\mu_m M^*)\wp(S^m e_m,e_m)\\ &\quad + (2+\mu_m M^*)\varkappa_m. \qquad \forall m \in N.\end{aligned} \tag{3.23}$$

By taking limit $m \to \infty$ and using (3.16), we get

$$\lim_{m\to\infty} \wp(S^m h_m, S^m e_m) = 0. \tag{3.24}$$

We have

$$\begin{aligned}\wp(S^m g_m, S^m h_m) &\leq \wp(g_m, h_m) + \mu_m \varphi \wp(g_m, h_m) + \varkappa_m.\\ &\leq (1+\mu_m M^*)\wp(g_m, h_m) + \varkappa_m.\\ &\leq (1+\mu_m M^*)\wp(S^m((1-\beta_m)h_m \oplus \beta_m S^m h_m), h_m) + \varkappa_m.\\ &\leq (1+\mu_m M^*)\wp(S^m((1-\beta_m)h_m \oplus \beta_m S^m h_m), S^m h_m)\\ &\quad + (1+\mu_m M^*)\wp(S^m h_m, h_m) + \varkappa_m.\\ &\leq (1+\mu_m M^*)[\wp((1-\beta_m)h_m \oplus \beta_m S^m h_m, h_m)\\ &\quad + \mu_m M^* \wp((1-\beta_m)h_m \oplus \beta_m S^m h_m, h_m) + \varkappa_m]\\ &\quad + (1+\mu_m M^*)\wp(S^m h_m, h_m) + \varkappa_m.\\ &\leq (1+\mu_m M^*)^2[\beta_m \wp(S^m h_m, h_m)] + (1+\mu_m M^*)\wp(S^m h_m, h_m)\\ &\quad + (2+\mu_m M^*)\varkappa_m. \qquad \forall m \in N.\end{aligned} \tag{3.25}$$

By taking limit $m \to \infty$ and using (3.22), we get

$$\lim_{m\to\infty} \wp(S^m g_m, S^m h_m) = 0. \tag{3.26}$$

From (3.16),(3.24) and (3.26), we get

$$\begin{aligned}\wp(e_m, e_{m+1}) &\leq \wp(e_m, S^m e_m) + \wp(S^m e_m, S^m h_m) + \wp(S^m h_m, S^m g_m)\\ &\to 0 \quad as \quad m \to \infty.\end{aligned} \tag{3.27}$$

Since S satisfies Definition 1.2 and uniformly L- Lipshitzian, we obtain

$$\begin{aligned}\wp(e_m, Se_m) &= \wp(e_m, e_{m+1}) + \wp(e_{m+1}, S^{m+1} e_{m+1})\\ &\quad + \wp(S^{m+1} e_{m+1}, S^{m+1} e_m) + \wp(S^{m+1} e_m, Se_m).\\ &\leq \wp(e_m, e_{m+1}) + \wp(e_{m+1}, S^{m+1} e_{m+1}) + L\wp(e_{m+1}, e_m)\\ &\quad + L\wp(s^m e_m, e_m)\\ &\to 0 \quad as \quad m \to \infty.\end{aligned} \tag{3.28}$$

Let $\breve{x} \in W_\Delta(e_m)$. Then, $\exists$ a subsequence $\{\breve{z}_m\}$ of $\{e_m\}$ such that $\mathcal{A}(\{\breve{z}_m\}) = \{\breve{x}\}$. By using Lemma 2.3, $\exists$ a subsequence $\{\breve{y}_m\}$ of $\{\breve{z}_m\}$ such that $\{\breve{y}_m\}$ Δ-converges to $\breve{y} \in \Theta$. By Lemma 2.4, $\breve{y} \in F(S)$. Since $\{\wp(\breve{z}_m, \breve{y})\}$ converges, by Lemma 2.2, $\breve{x} = \breve{y}$. This implies that $W_\Delta(e_m) \subseteq F(S)$. Next we will prove that $W_\Delta(e_m)$ consists of exactly one point. Let $\{\breve{z}_m\}$ be a subsequence of $\{e_m\}$ with $\mathcal{A}(\{\breve{z}_m\}) = \{\breve{x}\}$ and $\mathcal{A}(\{e_m\}) = \{e\}$. We have seen that $\breve{x} = \breve{y}$ and $\breve{y} \in F(S)$. Finally, since $\{\wp(e_m, \breve{y})\}$ converges, by Lemma 2.2, we have $e = \breve{y} \in F(S)$. This shows that $W_\Delta(e_m) = \{e\}$. □

Theorem 3.2. *Let M, S, Θ,(a),(b),(c),(d),(e), $\{\gamma_m\}$, $\{\beta_m\}$ and $\{\alpha_m\}$ same as in Theorem 3.1. Then, the sequence $\{e_m\}$, defined by (1.7) strongly converges to a fixed point of S iff*

$$\lim_{m\to\infty} inf\ \wp(e_m, F(S)) = 0.$$

where $\wp(\breve{x}, F(S)) = inf\{\wp(\breve{x}, \rho) \colon \rho \in F(S)\}$.

Proof. Necessity is obvious. Conversely, suppose that $\lim_{m\to\infty} inf\ \wp(e_m, F(S)) = 0$. As proved in Theorem 3.1, $\lim_{m\to\infty}\ \wp(e_m, F(S))$ exists for all $\rho \in F(S)$. Thus, by hypothesis, $\lim_{m\to\infty}\ \wp(e_m, F(S)) = 0$.

Next, we show that $\{e_m\}$ is cauchy sequence in C. Let $\epsilon > 0$ be arbitrary Chosen. Since $\lim_{m\to\infty}\ \wp(e_m, F(S)) = 0$, $\exists$ a positive integer m_0 such that for all $m \geq m_0$,

$$\wp(e_m, F(S)) < \frac{\epsilon}{4}.$$

In particular, $inf\{\wp(e_{m_0},\rho)\colon \rho \in F(S)\} < \frac{\epsilon}{2}$. Thus, $\exists\ \rho^* \in F(S)$ such that

$$\wp(e_{m_0},\rho^*) < \frac{\epsilon}{2}.$$

Now, for all $m, k \geq m_0$, we have

$$\wp(e_{m+k},e_m) \leq \wp(e_{m+k},\rho^*) + \wp(e_m,\rho^*) \leq 2\wp(e_{m_0},\rho^*) < 2(\frac{\epsilon}{2}) = \epsilon.$$

i.e., $\{e_m\}$ is a cauchy sequence in the closed subset Θ of a complete CAT(0) space and hence it converges to a point q in Θ. Now $\lim_{m\to\infty}\ \wp(e_m, F(S)) = 0$ gives that $\wp(q, F(S)) = 0$ and closedness of $F(S)$ forces q to be in $F(S)$. This completes the proof. □

Senter and Dotson [30] introduced a mapping satisfy the condition (I) as follows:

Definition 3.1. A mapping $S\colon \Theta \to \Theta$ is said to satisfy the Condition (I) ([30]) if $\exists$ a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(w) > 0$ for all $w \in (0,\infty)$ such that $\wp(e, Se) \leq f(\wp(e, F(S))$ for all $e \in \Theta$; where $\wp(\varsigma, F(S)) = inf\{\wp(e,\rho)\colon \rho \in F(S)\}$.

By using the similar technique as in the proof of Theorem 3.3 by Thakur et. al [1], we get the following result:

Theorem 3.3. *Let M, S, Θ,(a),(b),(c),(d),(e), $\{\gamma_m\}$, $\{\beta_m\}$ and $\{\alpha_m\}$ satisfy the hypothesis of theorem 3.1 and let S be a mapping satisfying Condition (I). Then the sequence $\{e_m\}$ generated by (1.7) converges strongly to a fixed point of S.*

Proof. As proved in theorem 3.2,$\lim_{m\to\infty} \wp(e_m, F(S))$ exists. Also, by the theorem 3.1, we have $\lim_{m\to\infty} \wp(e_m, Se_m) = 0$. It follows from Condition (I) that

$$\lim_{m\to\infty} f(\wp(e_m, F(S))) \leq \lim_{m\to\infty} \wp(e_m, Se_m) = 0.$$

That is, $\lim_{m\to\infty} f(\wp(e_m, F(S))) = 0$. Sience $f\colon [0,\infty) \to [0,\infty)$ is a non-decreasing function setisfying $f(0) = 0$ and $f(r) > 0$ for all $r > 0$, we obtain

$$\lim_{m\to\infty} f(\wp(e_m, F(S))) = 0.$$

Now all the conditions of Theorem 3.4 are satisfed, therefore by its conclusion $\{e_m\}$ converges strongly to a point of $F(S)$. □

Theorem 3.4. *Let M, S, Θ,(a),(b),(c),(d),(e), $\{\gamma_m\}$, $\{\beta_m\}$ and $\{\alpha_m\}$ be same as in Theorem 3.1 with S satisfies Condition(I). Then, $\{e_m\}$, defined by (1.7) converges to a point of $F(S)$.*

Proof. Recalling the definition of semi-compact mapping;
A map S defined on Θ is said to be semi-compact [31] if for a sequence $\{e_m\}$ in Θ with $\lim_{m\to\infty} \wp(e_m, Se_m) = 0$, $\exists$ a subsequence $\{e_{m_j}\}$ of $\{e_m\}$ such that $e_{m_j} \to \rho \in \Theta$.
By using the same steps used by Karapinar et al. [27] in the proof of Theorem 22, we get the next result. □

Corollary 3.1. Let M, S, Θ,(a),(b),(c),(d),(e), $\{\gamma_m\}$, $\{\beta_m\}$ and $\{\alpha_m\}$ be same as in Theorem 3.1. Let S be semi-compact. Then the sequence $\{e_m\}$ defined by (1.7) converges to a point of $F(S)$.

4 Numerical Example

In this part, we'll look at a total asymptotically non-expansive mapping as an example. **Example:**[32] Let $M = \mathbb{R}$ with usual metric and $C = [0,2]$. Let a self map S on Θ as follows:

$$Sx = \begin{cases} 1, & x \in [0,1], \\ \frac{1}{\sqrt{3}}\sqrt{4-x^2}, & x \in [1,2]. \end{cases}$$

Here S is a uniformly continuous and total asymptotically non-expansive mapping with $F(S) = \{1\}$. Also satisfies Condition (I), but S is not Lipschitzatian and hence it is not an asymptotically non-expansive mapping.

5 Conclusion

In this paper, we have presented a new type of iteration procedure for total asymptotically nonexpan-sive mapping in CAT(0) spaces. Our result generalizes, improve, extend and unify the results of Thakur et al. [1] and Izhar et al. [2].

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Thakur BS, Thakur D, Postolache M. Modified Picard-Mann hybrid iteration process for total asymptotically non-expansive mappings. Fixed Point Theory and Applications. 2015;2015:1-1.

[2] Uddin I, Khatoon S, Mlaiki N, Abdeljawad T. A modified iteration for total asymptotically non-expansive mappings in hadamard spaces. AIMS Mathematics. 2021;6(5):4758–4770.

[3] Mann WR. Mean value methods in iteration. Proceedings of the American Mathematical Society. 1953;4(3):506–510.

[4] Ishikawa S. Fixed points by a new iteration method. Proceedings of the American Mathematical Society. 1974;44(1):147-50.

[5] Noor MA. New approximation schemes for general variational inequalities. Journal of Mathematical Analysis and applications. 2000;251(1):217–229.

[6] Agarwal RP, O Regan D, Sahu D. Iterative construction of fixed points of nearly asymptotically non-expansive mappings. Journal of Nonlinear and convex Analysis. 2007;8(1):61.

[7] Kumam W, Pakkaranang N, Kumam P. Modified viscosity type iteration for total asymptotically non-expansive mappings in CAT (0) spaces and its application to optimization problems. J. Nonlinear Sci. Appl. 2018;11(2):288-302.

[8] Dhompongsa S, Panyanak B. On-convergence theorems in CAT (0) spaces. Computers Mathematics with Applications. 2008;56(10):2572-9.

[9] Goebel K, Kirk WA. A fixed point theorem for asymptotically non-expansive mappings. Proceedings of the American Mathematical Society. 1972;35(1):171-4.

[10] Bruhat F, Tits J. Groupes r´eductifs sur un corps local. Publications Math´ematiques de l'Institut des Hautes Etudes Scientifiques. 1972;41:5–251.

[11] Brown KS, Brown KS. Buildings. Springer; 1989.

[12] Bridson MR, Haefliger A. Metric Spaces of Non-Positive Curvature. Springer Science and Business Media. 2013;319.

[13] Basarir M, Sahin A. On the strong and Δ-convergence theorems for total asymptotically non-expansive mappings on CAT (0) space. Carpathian Mathematical Publications. 2013 Dec 30;5(2):170-9.

[14] Fukhar-ud-din H, Saleh K. One-step iterations for a finite family of generalized non-expansive mappings in CAT (0) spaces. Bulletin of the Malaysian Mathematical Sciences Society. 2018;41(2):597-608.

[15] Khatoon S, Uddin I, Pakkaranang N, Wairojjana N. Common fixed points of modified Picard-s iteration process involving two g-non-expansive mapping in CAT(0) space with directed graph. Indian J. Pure Appl. Math. 2019;18(1):1-13.

[16] Pakkaranang N, Kumam P. Strong and Δ-convergence for asymptotically K-strictly pseudo-contractive mappings in CAT(0) spaces. Communications in Mathematics and Applications. 2016;7(3):189–197.

[17] Bruck R, Kuczumow T, Reich S. Convergence of iterates of asymptotically non-expansive mappings in Banach spaces with the uniform Opial property. InColloquium Mathematicae. 1993;65(2):169-179.

[18] Gunduz B, Akbulut S. On the convergence of an iteration process for totally asymptotically I-non-expansive mappings. Journal of Nonlinear Analysis and Optimization: Theory & Applications. 2016;7(1):17-30.

[19] Alber Y, Chidume CE, Zegeye H. Approximating fixed points of total asymptotically non-expansive mappings. Fixed point theory and applications. 2006;2006:1-20.

[20] Chidume CE, Ofoedu EU. Approximation of common fixed points for finite families of total asymptotically non-expansive mappings. Journal of Mathematical Analysis and Applications. 2007;333(1):128-41.

[21] Chidume CE, Ali B. Approximation of common fixed points for finite families of nonself asymptotically non-expansive mappings in Banach spaces. Journal of Mathematical Analysis and Applications. 2007;326(2):960-73.

[22] Chang SS, Wang L, Lee HJ, Chan CK, Yang L. Demiclosed principle and -convergence theorems for total asymptotically non-expansive mappings in CAT (0) spaces. Applied Mathematics and Computation. 2012;219(5):2611-7.

[23] Plubtieng S, Ungchittrakool K. Strong convergence of modified ishikawa iteration for two asymptotically non-expansive mappings and semigroups. Nonlinear Analysis: Theory, Methods & Applications. 2007;43(1):2306–2315.

[24] Kim GE, Kim TH. Mann and Ishikawa iterations with errors for non-Lipschitzian mappings in Banach spaces. Computers & Mathematics with Applications. 2001;42(12):1565-70.

[25] Dhompongsa S, Kirk WA, Sims B. Fixed points of uniformly Lipschitzian mappings. Nonlinear analysis: theory, methods & applications. 2006;65(4):762-72.

[26] Dhompongsa S, Kirk WA, Panyanak B. non-expansive set-valued mappings in metric and Banach spaces. Journal of nonlinear and convex analysis. 2007;8(1):35-45.

[27] Karapınar E, Salahifard H, Vaezpour SM. Demiclosedness principle for total asymptotically non-expansive mappings in spaces. Journal of Applied Mathematics. 2014;22(3):353–355.

[28] Schu J. Weak and strong convergence to fixed points of asymptotically non-expansive mappings. Bulletin of the Australian Mathematical Society. 1991;43(1):153-9.

[29] Qihou L. Iterative sequences for asymptotically quasi-non-expansive mappings. Journal of Mathematical Analysis and applications. 2001;259(1):1-7.

[30] Senter HF, Dotson W. Approximating fixed points of non-expansive mappings. Proceedings of the American Mathematical Society. 1974;44(2):375-80.

[31] Shahzad N, Al-Dubiban R. A Approximating common fixed points of non-expansive mappings in Banach spaces. Science Asia. 2006;41:289–294.

[32] Kim GE. Strong convergence to a fixed point of a total asymptotically non-expansive mapping. Fixed Point Theory and Applications. 2013;2013(1):1-0.

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