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A Certain Subclass of Multivalent Analytic Functions Defined by Fractional Calculus Operator

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Abstract

In this paper we introduce a new subclass of multivalent analytic functions defined by fractional calculus operator. Such results as subordination and superordination properties, convolution properties, inequality properties and other interesting properties of this subclass are proved.

Keywords: Analytic function, multivalent function, fractional differintegral operator, convex univalent function, hadamard product, subordination, superordination.

1 Introduction

Let $H(U)$ be the class of functions analytic in $U=\{z: z\in C \text{ and } |z|<1\}$ and $H[a,k]$ be the subclass of $H(U)$ consisting of functions of the form $f(z)=a+a_k z^k+a_{k+1}z^{k+1}+...$, with $H_0\equiv H[0,1]$ and $H\equiv H[1,1]$.

Let $A_p(k)$ denote the class of functions of the form

$$f(z)=z^p+\sum_{n=k}^{\infty}a_{n+p}z^{n+p} \quad \left(p,k\in \mathrm{N}=\{1,2,3,...\};z\in U\right), \tag{1.1}$$

which are analytic in the open unit disk U , and let $A_p(1)=A_p$ and $A_1(1)=A$.

A function $f(z)\in A_p(k)$ of the form (1.1) is said to be in the class $S^*_{p,k}(\rho)$ of multivalent (p-valent) starlike of order ρ $\left(0\le\rho<p\right)$, if it satisfies the following inequality:

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$$\Re\left\{\frac{z f'(z)}{f(z)}\right\} > \rho \qquad (0 \le \rho < p, z \in U). \tag{1.2}$$

Let f and F be members of $H(U)$, the function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is said to be superordinate to $f(z)$, if there exists a function $w(z)$ analytic in U with $w(0)=0$ and $|w(z)|<1$ $(z \in U)$, such that $f(z)=F(w(z))$. In such a case we write $f(z) \prec F(z)$. In particular, if F is univalent, then $f(z) \prec F(z)$ if and only if $f(0)=F(0)$ and $f(U) \subset F(U)$ [1,2].

For two functions $f(z)$ given by (1.1) and

$$g(z)=z^p+\sum_{n=k}^{\infty} b_{n+p} z^{n+p}, \tag{1.3}$$

The hadmard product (or convolution) of f and g is defined by

$$(f*g)(z)=z^p+\sum_{n=k}^{\infty} a_{n+p} b_{n+p} z^{n+p}=(g*f)(z). \tag{1.4}$$

We recall the definitions of the fractional derivative and integral operators introduced and studied by Saigo (cf.[16] and [21], see also [18,19,20 and 22]).

Definition 1.1 Let $\alpha>0$ and $\beta,\gamma \in R$, then the generalized fractional integral operator $I_{0,z}^{\alpha,\beta,\gamma}$ of order α of a function $f(z)$ is defined by function $f(z)$

$$I_{0,z}^{\alpha,\beta,\gamma} f(z)=\frac{z^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta,-\gamma;\alpha;1-\frac{t}{z}\right) f(t)\,dt, \tag{1.5}$$

where the function $f(z)$ is analytic in a simply -connected region of the z −plane containing the origin and where the multiplicity of $(z-t)^{\alpha-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t)>0$ provided further that

$$f(z)=O\left(|z|^{\varepsilon}\right), z \to 0 \; for \; \varepsilon > \max(0,\beta-\gamma)-1. \tag{1.6}$$

Definition 1.2 Let $0 \le \alpha < 1$ and $\beta,\gamma \in R$, then the generalized fractional derivative operator $J_{0,z}^{\alpha,\beta,\gamma}$ of order α of a function $f(z)$ is defined by

$$J_{0,z}^{\alpha,\beta,\gamma} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz}\left[z^{\alpha-\beta} \int_0^z (z-t)^{-\alpha} \, {}_2F_1\left(\beta-\alpha, 1-\gamma; 1-\alpha; 1-\frac{t}{z}\right) f(t)\,dt \right], \quad (1.7)$$

$$= \frac{d^n}{dz^n} J_{0,z}^{\alpha,\beta,\gamma} f(z) \qquad (n \le \alpha < n+1; n \in N).$$

where the function $f(z)$ be analytic in a simply -connected region of the $z-$plane containing the origin with the order as given in (1.6) and the multiplicity of $(z-t)^{\alpha}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Note that

$$I_{0,z}^{\alpha,-\alpha,\gamma} f(z) = D_z^{-\alpha} f(z), \quad (\alpha > 0) \qquad (1.8)$$

$$J_{0,z}^{\alpha,\alpha,\gamma} f(z) = D_z^{\alpha} f(z), \quad (0 \le \alpha < 1) \qquad (1.9)$$

where $D_z^{-\alpha} f(z)$ and $D_z^{\alpha} f(z)$ are respectively the known Riemann- Liouvill fractional integral and derivative operator (cf. [13] and [14], see also [25]).

Definition 1.3 For real number $\alpha(-\infty < \alpha < 1)$ and $\beta(-\infty < \beta < 1)$ and a positive real number γ, the fractional operator $U_{0,z}^{\alpha,\beta,\gamma} : A_p \to A_p$ is defined in terms of $J_{0,z}^{\alpha,\beta,\gamma}$ and $I_{0,z}^{\alpha,\beta,\gamma}$ by [12,5]

$$U_{0,z}^{(\alpha,\beta,\gamma)} f(z) = z^p + \sum_{n=1}^{\infty} \frac{(1+p)_n (1+p+\gamma-\beta)_n}{(1+p-\beta)_n (1+p+\gamma-\alpha)_n} a_{n+p} z^{n+p}, \qquad (1.10)$$

which for $f(z) \ne 0$ may be written as

$$U_{0,z}^{\alpha,\beta,\gamma} f(z) = \begin{cases} \frac{\Gamma(1+p-\beta)\Gamma(1+p+\gamma-\alpha)}{\Gamma(1+p)\Gamma(1+p+\gamma-\beta)} z^{\beta} J_{0,z}^{\alpha,\beta,\gamma} f(z); & 0 \le \alpha \le 1 \\ \frac{\Gamma(1+p-\beta)\Gamma(1+p+\gamma-\alpha)}{\Gamma(1+p)\Gamma(1+p+\gamma-\beta)} z^{\beta} I_{0,z}^{-\alpha,\beta,\gamma} f(z); & -\infty \le \alpha \le 0 \end{cases} \qquad (1.11)$$

Where $J_{0,z}^{\alpha,\beta,\gamma} f(z)$ and $I_{0,z}^{-\alpha,\beta,\gamma} f(z)$ are, respectively the fractional derivative of f of order α if $0 \le \alpha < 1$ and the fractional integral of f of order $-\alpha$ if $-\infty \le \alpha < 0$.

It is easily verified (see Choi [4]) from (1.10) that

$$\left(p-\beta\right)U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f\left(z\right)+\beta U_{0,z}^{\alpha+1,\beta+1,\gamma+1}f\left(z\right)=z\left(U_{0,z}^{\alpha,\beta,\gamma}f\left(z\right)\right)'. \tag{1.12}$$

Note that

$$U_{0,z}^{\alpha,\beta,\gamma}f\left(z\right)=\Omega_z^{(\alpha,p)}f\left(z\right)\qquad\left(-\infty<\alpha<1\right), \tag{1.13}$$

The fractional differintegral operator $\Omega_z^{(\alpha,p)}f\left(z\right)$ and $\left(-\infty<\alpha<p+1\right)$ is studied Patel and Mishra [15], and the fractional differential operator $\Omega_z^{(\alpha,p)}$ with $0<\alpha<1$ was investigated by Srivastava and Aouf [26]. We, further observe that $\Omega_z^{(\alpha,1)}=\Omega_z^{\alpha}$ is the operator introduced and studied by Owa and Srivastava [14].

It is interesting to observe that

$$U_{0,z}^{0,0,\gamma}f\left(z\right)=f\left(z\right) \tag{1.14}$$

$$U_{0,z}^{1,1,\gamma}f\left(z\right)=\frac{z}{p}f'\left(z\right) \tag{1.15}$$

By making use of the differintegral operator $\Omega_z^{(\alpha,p)}$ and the above mentioned principle of subordination between analytic functions, we introduce and investigate the following subclass of the class o $A_p(k)$ f p -valent analytic functions.

Definition 1.4 A function $f\left(z\right)\in A_p(k)$ is said to be in the class $S_{p,k}^{\lambda,\mu}\left(\alpha;A,B\right)$ if it satisfies the following subordination condition:

$$\left(1-\lambda\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}{z^p}\right)^{\mu}+\lambda\left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)}f\left(z\right)}{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}{z^p}\right)^{\mu}\prec\frac{1+Az}{1+Bz}, \tag{1.16}$$

$(-\infty<\alpha<p;\ -1\le B<A\le 1;\ A\ne B;A\in\sqcup;\ p,k\in\ ;\ \lambda\in\quad \text{and } \mathrm{Re}(\mu)>0)$.

It may be noted that for suitable choice of μ,A,B,p,λ and α the class $S_{p,k}^{\lambda,\mu}\left(\alpha;A,B\right)$ extends several classes of analytic and p -valent functions studied by several authors such as Aouf and Seoudy [3], Shenan [24], Yang [27], Zhou and Owa [28] and Liu [6].

To prove our results, we need the following definitions and lemmas.

Definition 1.5 ([10]). Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\bar{U}/E(f)$ where

$$E(f)=\{\zeta \in \partial U : \lim_{z\to\zeta} f(z)=\infty\},$$

and are such that $f'(\zeta)\neq 0$ for $\zeta \in \partial U / E(f)$.

Lemma 1.1 ([11]). Let the function $h(z)$ be analytic and convex (univalent) in U with $h(0)=1$. Suppose also that the function $g(z)$ given by

$$g(z)=1+c_k z^k + c_{k+1} z^{k+1}+... \tag{1.17}$$

is analytic in U. If

$$g(z)+\frac{zg'(z)}{\gamma} \prec h(z) \quad \left(\Re(\gamma)>0; \gamma\neq 0; z\in U\right), \tag{1.18}$$

Then

$$g(z)\prec q(z)=\frac{\gamma}{k} z^{-\frac{\gamma}{k}} \int h(t) t^{\frac{\gamma}{k}} dt \prec h(t),$$

and $q(z)$ is the best dominant of (1.18).

Lemma 1.2 ([23]). Let $q(z)$ be a convex univalent function in U and let $\alpha \in \sqcup$, $\eta \in \sqcup^{*} = \ \backslash\{0\}$ with

$$\Re\left\{1+\frac{z\,q''(z)}{q'(z)}\right\} > \max\left\{0, -\Re\left(\frac{\sigma}{\eta}\right)\right\}.$$

If the function $g(z)$ is analytic in U and

$$\sigma g(z)+\eta z g'(z) \prec \sigma q'(z)+\eta z q'(z)\ ,$$

then $g(z)\prec q(z)$ and $q(z)$ is the best dominant.

Lemma 1.3 ([11]). Let $q(z)$ be convex univalent function in U and let $k \in \sqcup$. Further assume $\mathrm{Re}(k)>0$. If $g(z)\in H[q(0),1]\bigcap Q$, and $g(z)+kzg'(z)$ is univalent in U, then

$$q(z)+kzq'(z)\prec g(z)+kzg'(z),$$

implies $g(z)\prec q(z)$ and $q(z)$ is the best subordinate.

Lemma 1.4 ([25]). Let the function F be analytic and convex in U. If $f, g\in A$ and $f, g\prec F$, then $\lambda f+(1-\lambda)g\prec F \quad (0\le\lambda\le 1)$.

Lemma 1.5 ([17]). Let $f(z)=1+\sum_{k=1}^{\infty}a_k z^k$, be analytic in U and $g(z)=1+\sum_{k=1}^{\infty}b_k z^k$ be analytic and convex in U. If $f(z)\prec g(z)$, then

$$|a_k|<|b_1| \qquad (k\in \mathbb{N}).$$

Lemma 1.6 ([8]). Let $0\neq\delta\in\mathbb{N}$, $\frac{p}{\delta}>0,\ 0\le\rho<1,\ g(z)\in H[1,k]$ and

$$g(z)\prec +Lz \qquad \left(L=\frac{\nu M}{k\delta+\nu}\right),$$

where

$$M=M_k(\delta,\nu,\rho)=\frac{(1-\rho)|\delta|\left(1+\frac{k\delta}{\nu}\right)}{|1-\delta+\rho\delta|+\sqrt{1+\left(1+\frac{k\delta}{\nu}\right)^2}}.$$

If $h(z)\in H[1,k]$ satisfies the following subordination condition;

$$g(z)\left[1-\delta+\delta(1-\rho)h(z)+\rho\right]\prec 1+Mz,$$

then

$$\Re(h(z))>0 \quad (z\in U).$$

2 Main Result

Theorem 2.1 Let $f(z)\in S_{p,k}^{\lambda,\mu}(\alpha;A,B)$ with $\Re(\lambda)>0$. Then

$$\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu} \prec q(z)=\frac{(p-\beta)\mu}{\lambda k}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{(p-\beta)\mu}{\lambda k}-1}du \prec \frac{1+Az}{1+Bz}, \tag{2.1}$$

and $q(z)$ is the best dominant.

Proof. Define the function $g(z)$ by

$$g(z)=\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu} \quad (z\in U). \tag{2.2}$$

Then $g(z)$ is of the form (1.17) and analytic in U. Differentiating (2.1) with respect to z and using (1. 12), we get

$$(1-\lambda)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}+\lambda\left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)}f(z)}{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}=g(z)+\frac{\lambda zg'(z)}{(p-\beta)\mu}\prec\frac{1+Az}{1+Bz}. \tag{2.3}$$

Applying Lemma 1.1 to (2.3) with $\gamma=\dfrac{(p-\beta)\mu}{\lambda}$, we get

$$\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu} \prec q(z)=\frac{(p-\beta)\mu}{\lambda k}\int_0^z \frac{1+At}{1+Bt}t^{\frac{(p-\beta)\mu}{\lambda k}-1}dt$$

$$=\frac{(p-\beta)\mu}{\lambda k}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{(p-\beta)\mu}{\lambda k}-1}du \prec \frac{1+Az}{1+Bz}, \tag{2.4}$$

and $q(z)$ is the best dominant.

Theorem 2.2 Let $q(z)$ be univalent function in U and let $\lambda\in\mathbb{C}^*$. Suppose also that $q(z)$ satisfies the following inequality:

$$\Re\left\{1+\frac{z\,q''(z)}{q'(z)}\right\}>\max\left\{0,-\Re\left(\frac{(p-\beta)\mu}{\lambda}\right)\right\}. \tag{2.5}$$

If $f\in A_p$ satisfies the following subordination:

$$(1-\lambda)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}{z^{p}}\right)^{\mu}+\lambda\left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)}f\left(z\right)}{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}{z^{p}}\right)^{\mu}\prec q(z)+\frac{\lambda z q'(z)}{(p-\beta)\mu}, \quad (2.6)$$

Then

$$\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}{z^{p}}\right)^{\mu}\prec q(z),$$

and $q(z)$ is the best dominant.

Proof. Let the function $g(z)$ be defined by (2.2). we know that (2.3) holds true. Combining (2.3) and (2.6), we find that

$$g(z)+\frac{\lambda z g'(z)}{(p-\beta)\mu}\prec q(z)+\frac{\lambda z q'(z)}{(p-\beta)\mu}. \quad (2.7)$$

By using Lemma 1.2 and (2.7), we easily get the assertion of Theorem 2.2.

Taking $q(z)=\frac{1+Az}{1+Bz}$ in Theorem 2.2, we get the following result.

Corollary 2.1 Let $\lambda \in \mathbb{C}^*$ and $-1 \le B < A \le 1$. Suppose also that

$$\Re\left\{\frac{1-Bz}{1+Bz}\right\} > \max\left\{0, -\Re\left(\frac{(p-\beta)\mu}{\lambda}\right)\right\}.$$

If $f \in A_p$ satisfies the following subordination:

$$(1-\lambda)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}{z^{p}}\right)^{\mu}+\lambda\left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)}f\left(z\right)}{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}{z^{p}}\right)^{\mu}\prec\frac{1+Az}{1+Bz}+\frac{\lambda}{(p-\beta)\mu}\frac{(A-B)z}{(1+Bz)^{2}},$$

then

$$\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}{z^{p}}\right)^{\mu}\prec\frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Theorem 2.3 Let $q(z)$ be convex univalent function in U and let $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0.$ Also let

$$\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu} \in H[q(0),1] \cap Q ,$$

and

$$(1-\lambda)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)}f(z)}{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}$$

be univalent in U. If

$$q(z) + \frac{\lambda z q'(z)}{(p-\beta)\mu} \prec (1-\lambda)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{\Omega_{z}^{(\alpha+1,\beta+1,\gamma+1)}f(z)}{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu} ,$$

then

$$q(z) \prec \left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu},$$

and $q(z)$ is the best subordinate.

Proof. Let the function $g(z)$ be defined by (2.2). Then

$$q(z) + \frac{\lambda z q'(z)}{(p-\alpha)\mu} \prec (1-\lambda)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)}f(z)}{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}$$

$$= g(z) + \frac{\lambda z g'(z)}{(p-\beta)\mu}.$$

By using Lemma 1.3 we easily get the assertion of theorem 2.3.

Taking $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 2.3, we get the following result.

Corollary 2.2 Let $q(z)$ be convex univalent function in U and $-1 \le B < A \le 1$, $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0.$ Also let

$$0 \neq \left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)} f(z)}{z^p} \right)^{\mu} \in H[q(0),1] \cap Q ,$$

and

$$(1-\lambda)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)} f(z)}{z^p} \right)^{\mu} + \lambda \left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)} f(z)}{U_{0,z}^{(\alpha,\beta,\gamma)} f(z)} \right) \left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)} f(z)}{z^p} \right)^{\mu}$$

be univalent in U. If

$$\frac{1+Az}{1+Bz} + \frac{\lambda}{(p-\beta)\mu} \frac{(A-B)z}{(1+Bz)^2} \prec (1-\lambda)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)} f(z)}{z^p} \right)^{\mu} + \lambda \left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)} f(z)}{U_{0,z}^{(\alpha,\beta,\gamma)} f(z)} \right) \left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)} f(z)}{z^p} \right)^{\mu} ,$$

Then

$$\frac{1+Az}{1+Bz} \prec \left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)} f(z)}{z^p} \right)^{\mu} ,$$

and $\dfrac{1+Az}{1+Bz}$ is the best subordinate.

Combining the above results of subordination and superordination, we easily get the following "sandwich-type result".

Corollary 2.3 Let $q_1(z)$ be convex function in U and let $q_2(z)$ be univalent function in U, let $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0.$ let $q_2(z)$ satisfy (2.5). If

$$0 \neq \left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)} f(z)}{z^p} \right)^{\mu} \in H[q(0),1] \cap Q ,$$

and

$$(1-\lambda)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)} f(z)}{z^p} \right)^{\mu} + \lambda \left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)} f(z)}{U_{0,z}^{(\alpha,\beta,\gamma)} f(z)} \right) \left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)} f(z)}{z^p} \right)^{\mu}$$

is univalent in U , and also

$$q_1(z)+\frac{\lambda z q_1'(z)}{(p-\beta)\mu} \prec (1-\lambda)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}+\lambda\left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)}f(z)}{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}$$

$$\prec q_2(z)+\frac{\lambda z q_2'(z)}{(p-\beta)\mu},$$

then

$$q_1(z)\prec\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}\prec q_2(z),$$

and $q_1(z)$ and $q_2(z)$ are respectively, the best subordinate and dominant.

Theorem 2.4 If $\lambda,\ \mu>0$ and $f(z)\in S_{p,k}^{0,\mu}(\alpha;1-2\rho,-1)$ $(0\le\rho<1)$, then $f(z)\in S_{p,k}^{\lambda,\mu}(\alpha;1-2\rho,-1)$ for $|z|<R$, where

$$R=\left(\sqrt{\left(\frac{\lambda k}{(p-\alpha)\mu}\right)^2+1}-\frac{\lambda k}{(p-\alpha)\mu}\right)^{\frac{1}{k}}. \tag{2.8}$$

The bound R is the best possible.

Proof. We begin by writing

$$\left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)}f(z)}{z^p}\right)^{\mu}=\rho+(1-\rho)g(z) \qquad (z\in U;0\le\rho<1). \tag{2.9}$$

Then, clearly, the function $g(z)$ is of the form (1.17), is analytic and has a positive real part in U. Differentiating (2.9) with respect to z and using the identity (1.12), we get

$$\frac{1}{1-\rho}\left\{(1-\lambda)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}+\lambda\left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)}f(z)}{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}-\rho\right\}$$

$$=g(z)+\frac{\lambda z g'(z)}{(p-\beta)\mu}. \tag{2.10}$$

By making use of the following well-known estimate (see [9]):

$$\frac{|zg'(z)|}{\Re(g(z))} \leq \frac{2kr^k}{1-r^{2k}} \qquad \left(|z| < r < 1\right)$$

In (2.10), we obtain that

$$\Re\left(\frac{1}{1-\rho}\left\{(1-\lambda)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)}f(z)}{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu} - \rho\right\}\right)$$

$$\geq \Re\{g(z)\}\left(1 - \frac{2kr^k\lambda}{(p-\beta)\mu\left(1-r^{2k}\right)}\right). \tag{2.11}$$

It is seen that the right-hand side of (2.11) is positive, provided that $r < R$, where R is given by (2.8).

In order to show that the bound R is the best possible, we consider the function $f(z) \in A_p(k)$ defined by

$$\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu} = \rho + (1-\rho)\frac{1+z^k}{1-z^k} \qquad \left(z \in U; 0 \leq \rho < 1\right).$$

Noting that

$$\frac{1}{1-\rho}\left\{(1-\lambda)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)}f(z)}{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu} - \rho\right\}$$

$$= \frac{1+z^k}{1-z^k} + \frac{2k\lambda z^k}{(p-\beta)\mu\left(1+z^k\right)^2} = 0. \tag{2.12}$$

for $|z| < R$, we conclude that the bound is the best possible. Theorem 2.4 is thus proved.

Theorem 2.5 Let $f(z) \in S_{p,k}^{\lambda,\mu}\left(\alpha; A, B\right)$ with $\Re(\lambda) > 0$. Then

$$f(z) = \left(z^p\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\frac{1}{\mu}}\right) * \left(z^p + \sum_{n=1}^{\infty}\frac{\left(1+p-\beta\right)_n\left(1+p+\gamma-\alpha\right)_n}{\left(1+p\right)_n\left(1+p+\gamma-\beta\right)_n}z^{n+p}\right) \tag{2.13}$$

where $w(z)$ is an analytic function with $w(0)=0$ and $|w(z)|<1$ $(z\in U)$.

Proof. Let $f(z)\in S_{p,k}^{\lambda,\mu}(\alpha;A,B)$ with $\Re(\lambda)>0.$ It follows from (2.1) that

$$\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}=\frac{1+Aw(z)}{1+Bw(z)}, \tag{2.14}$$

where $w(z)$ is an analytic function with $w(0)=0$ and $|w(z)|<1$ $(z\in U)$. By virtue of (2.14), we easily find that

$$U_{0,z}^{(\alpha,\beta,\gamma)}f(z)=z^p\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\frac{1}{\mu}}\ . \tag{2.15}$$

Combining (1.9) and (2.15), we have

$$\left(z^p+\sum_{n=1}^{\infty}\frac{(1+p)_n(1+p+\gamma-\beta)_n}{(1+p-\beta)_n(1+p+\gamma-\alpha)_n}z^{n+p}\right)*f(z)=\left(z^p\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\frac{1}{\mu}}\right) \tag{2.16}$$

The assertion (2.13) of Theorem 2.5 can now easily be derived from (2.16).

Theorem 2.6 Let $f(z)\in S_{p,k}^{\lambda,\mu}(\alpha;A,B)$ with $\Re(\lambda)>0.$ Then

$$\frac{1}{z^p}\left[\left(1+Be^{i\theta}\right)^{\frac{1}{\mu}}\left(z^p+\sum_{n=1}^{\infty}\frac{(1+p)_n(1+p+\gamma-\beta)_n}{(1+p-\beta)_n(1+p+\gamma-\alpha)_n}z^{n+p}\right)*f(z)\right.$$
$$\left.-z^p\left(1+Ae^{i\theta}\right)^{\frac{1}{\mu}}\right]\neq 0\quad (z\in U;\ 0<\theta<2\pi)\ . \tag{2.17}$$

Proof. Let $f(z)\in S_{p,k}^{\lambda,\mu}(\alpha;A,B)$ with $\Re(\lambda)>0.$ We know that (2.1) holds true, which implies that

$$\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}\neq\frac{1+Ae^{i\theta}}{1+Be^{i\theta}}\qquad (z\in U;\ 0<\theta<2\pi)\ . \tag{2.18}$$

It is easy to see that the condition (2.18) can be written as follows:

$$\frac{1}{z^p}\left[U_{0,z}^{(\alpha,\beta,\gamma)}f(z)\left(1+Be^{i\theta}\right)^{\frac{1}{\mu}}-z^p\left(1+Ae^{i\theta}\right)^{\frac{1}{\mu}}\right]\neq 0 \quad (z\in U;\ 0<\theta<2\pi). \tag{2.19}$$

Combining (1.10) and (2.19), we easily get the convolution property (2.17) asserted by Theorem 2.6.

Theorem 2.7 Let $\lambda_2 \geq \lambda_1 \geq 0$ and $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$. Then

$$S_{p,k}^{\lambda_2,\mu}\left(\alpha;A_2,B_2\right)\subset S_{p,k}^{\lambda_1,\mu}\left(\alpha;A_1,B_1\right). \tag{2.20}$$

Proof. Let $f(z)\in S_{p,k}^{\lambda_2,\mu}\left(\alpha;A_2,B_2\right)$. Then

$$\left(1-\lambda_2\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}+\lambda_2\left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)}f(z)}{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}\prec\frac{1+A_2z}{1+B_2z}.$$

Since $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, we easily find that

$$\left(1-\lambda_2\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}+\lambda_2\left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)}f(z)}{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}\prec\frac{1+A_2z}{1+B_2z}\prec\frac{1+A_1z}{1+B_1z}, \tag{2.21}$$

that is $f(z)\in S_{p,k}^{\lambda_2,\mu}\left(\alpha;A_1,B_1\right)$. Thus the assertion (2.20) holds for $\lambda_2=\lambda_1\geq 0$. If $\lambda_2\geq\lambda_1\geq 0$, by Theorem 2.1 and (2.21), we know that $f(z)\in S_{p,k}^{\lambda_0,\mu}\left(\alpha;A_1,B_1\right)$, that is

$$\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}\prec\frac{1+A_1z}{1+B_1z}, \tag{2.22}$$

At the same time, we have

$$\left(1-\lambda_1\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}+\lambda_1\left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)}f(z)}{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}$$

$$=\left(1-\frac{\lambda_1}{\lambda_2}\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}+\frac{\lambda_1}{\lambda_2}\left[\left(1-\lambda_2\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}+\lambda_2\left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)}f(z)}{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}\right]. \tag{2.23}$$

Moreover, $0 \le \frac{\lambda_1}{\lambda_2} < 1$, and the function $\frac{1+A_1 z}{1+B_1 z}$ $\left(-1 \le B_1 < A_1 \le 1; z \in U\right)$ is analytic and convex in U . Combining (2.21)-(2.23) using Lemma 1.4, we find that

$$\left(1-\lambda_1\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}{z^p}\right)^{\mu} + \lambda_1\left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)}f\left(z\right)}{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}{z^p}\right)^{\mu} \prec \frac{1+A_1 z}{1+B_1 z},$$

that is $f(z) \in S_{p,k}^{\lambda_1,\mu}\left(\alpha; A_1, B_1\right)$, which implies that the assertion (2.20) of Theorem 2.7 holds.

Theorem 2.8 Let $f(z) \in S_{p,k}^{\lambda,\mu}\left(\alpha; A, B\right)$ with $\Re(\lambda) > 0$ and $-1 \le B_1 < A_1 \le 1$. Then

$$\frac{(p-\beta)\mu}{\lambda k}\int_0^1 \frac{1-Au}{1-Bu} u^{\frac{(p-\beta)\mu}{\lambda k}-1} du$$
$$< \Re\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}{z^p}\right)^{\mu} < \frac{(p-\beta)\mu}{\lambda k}\int_0^1 \frac{1+Au}{1+Bu} u^{\frac{(p-\beta)\mu}{\lambda k}-1} du. \qquad (2.24)$$

The extremal function of (2.24) is defined by

$$U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right) = z^p\left(\frac{(p-\beta)\mu}{\lambda k}\int_0^1 \frac{1+Azu}{1+zBu} u^{\frac{(p-\beta)\mu}{\lambda k}-1} du\right)^{\frac{1}{\mu}}. \qquad (2.25)$$

Proof. Let $f(z) \in S_{p,k}^{\lambda,\mu}\left(\alpha; A, B\right)$ with $\Re(\lambda) > 0.$ From Theorem 2.1 we know that (2.1) holds true, which implies that

$$\Re\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}{z^p}\right)^{\mu} < \sup_{z\in U}\Re\left\{\frac{(p-\beta)\mu}{\lambda k}\int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{(p-\beta)\mu}{\lambda k}-1} du\right\}$$
$$\le \frac{(p-\beta)\mu}{\lambda k}\int_0^1 \sup_{z\in U}\Re\left(\frac{1+Azu}{1+Bzu}\right) u^{\frac{(p-\beta)\mu}{\lambda k}-1} du$$
$$\le \frac{(p-\beta)\mu}{\lambda k}\int_0^1 \frac{1+Au}{1+Bu} u^{\frac{(p-\beta)\mu}{\lambda k}-1} du, \qquad (2.26)$$

and

$$\Re\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^{p}}\right)^{\mu} > \inf_{z\in U}\Re\left\{\frac{(p-\beta)\mu}{\lambda k}\int_{0}^{1}\frac{1+Azu}{1+Bzu}u^{\frac{(p-\beta)\mu}{\lambda k}-1}du\right\}$$

$$\geq \frac{(p-\beta)\mu}{\lambda k}\int_{0}^{1}\inf_{z\in U}\Re\left(\frac{1+Azu}{1+Bzu}\right)u^{\frac{(p-\beta)\mu}{\lambda k}-1}du$$

$$\geq \frac{(p-\beta)\mu}{\lambda k}\int_{0}^{1}\frac{1+Au}{1+Bu}u^{\frac{(p-\beta)\mu}{\lambda k}-1}du, \tag{2.27}$$

Combining (2.26) and (2.27), we get (2.24). By noting that the function $U_{0,z}^{(\alpha,\beta,\gamma)}f(z)$ defined by (2.25) belongs to the class $S_{p,k}^{\lambda,\mu}(\alpha;A,B)$, we obtain that equality (2.24) is sharp. The proof of Theorem 2.8 is evidently completed.

In view of Theorem 2.8, we easily derive the following distortion theorems for the class $S_{p,k}^{\lambda,\mu}(\alpha;A,B)$.

Corollary 2.4 Let $f(z)\in S_{p,k}^{\lambda,\mu}(\alpha;A,B)$ with $\Re(\lambda)>0$ and $-1\leq B_1<A_1\leq 1$. Then for $|z|=r<1$, we have

$$r^{p}\left(\frac{(p-\beta)\mu}{\lambda k}\int_{0}^{1}\frac{1-Aur}{1-Bur}u^{\frac{(p-\beta)\mu}{\lambda k}-1}du\right)^{\frac{1}{\mu}}$$

$$<\left|U_{0,z}^{(\alpha,\beta,\gamma)}f(z)\right|<r^{p}\left(\frac{(p-\beta)\mu}{\lambda k}\int_{0}^{1}\frac{1+Aur}{1+Bur}u^{\frac{(p-\beta)\mu}{\lambda k}-1}du\right)^{\frac{1}{\mu}}. \tag{2.28}$$

The extremal function of (2.28) is defined by (2.25).

By noting that

$$\left(\Re(\nu)\right)^{\frac{1}{2}}\leq\Re\left(\nu^{\frac{1}{2}}\right)\leq\left|\nu^{\frac{1}{2}}\right| \quad \left(\nu\in\sqcup;\Re(\nu)\geq 0\right).$$

From Theorem 2.8, we easily get the following results.

Corollary 2.5 Let $f(z)\in S_{p,k}^{\lambda,\mu}(\alpha;A,B)$ with $\Re(\lambda)>0$ and $-1\leq B_1<A_1\leq 1$. Then

$$\left(\frac{(p-\beta)\mu}{\lambda k}\int_{0}^{1}\frac{1-Au}{1-Bu}u^{\frac{(p-\beta)\mu}{\lambda k}-1}du\right)^{\frac{1}{2}}$$

$$< \Re\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\frac{\mu}{2}} < \left(\frac{(p-\beta)\mu}{\lambda k}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{(p-\beta)\mu}{\lambda k}-1}du\right)^{\frac{1}{2}}.$$

Theorem 2.9 Let $f(z)$ defined by (1.1) be in the class $S_{p,k}^{\lambda,\mu}(\alpha;A,B)$, Then

$$\left|a_{n+p}\right| \le \frac{(1+p-\beta)_k(1+p+\gamma-\alpha)_k}{(1+p)_k(1+p+\gamma-\beta)_k}\left|\frac{A-B}{\lambda k+\mu(p-\beta)}\right| \tag{2.29}$$

The inequality (2.29) is sharp, with the extremal function defined by (2.25)

.

Proof. Combining (1.1) and (1.16), we obtain

$$(1-\lambda)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu} + \lambda\left(\frac{U_{0,z}^{(\alpha,+1,\beta+1,\gamma+1)}f(z)}{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f(z)}{z^p}\right)^{\mu}$$

$$=1+\left[\lambda k+\mu(p-\beta)\right]\frac{(1+p+\gamma-\beta)_k(1+p)_k}{(1+p-\beta)_k(1+p+\gamma-\alpha)_k}a_{p+k}z^k+\ldots$$

$$\prec \frac{1+Az}{1+Bz} = 1+(A-B)z+\ldots \tag{2.30}$$

An application of Lemma 1.5 to (2.30) yields

$$\frac{(1+p+\gamma-\beta)_k(1+p)_k}{(1+p-\beta)_k(1+p+\gamma-\alpha)_k}\left|\left[\lambda k+\mu(p-\beta)\right]a_{n+p}\right| \le |A-B|\cdot \tag{2.31}$$

Thus, from (2.31), we easily arrive at (2.29) asserted by Theorem 2.9.

Theorem 2.10 Let $0 \ne \lambda \in \square$, $\mu \in$, $\alpha > 1$, $\frac{\mu}{\lambda} > 0$ and $0 \le \rho < 1$. If $f(z) \in S_{p,k}^{\lambda,\mu}(\alpha;A,0)$ with

$$A = \frac{(1-\rho)|\lambda|\left(1+\frac{k\lambda}{(P-\beta)\mu}\right)}{|1-\lambda+\rho\lambda|+\sqrt{1+\left(1+\frac{k\lambda}{(P-\beta)\mu}\right)^2}},$$

then

$$\Omega_z^{(\alpha,p)}f(z) \in S_{p,k}^{*}(p\rho-(p-\beta)(1-\rho)).$$

Proof. Suppose that $f(z) \in S_{p,k}^{\lambda,\mu}(\alpha;A,0)$. By (1.16), we have

$$\left(1-\lambda\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}{z^{p}}\right)^{\mu}+\lambda\left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)}f\left(z\right)}{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}\right)\left(\frac{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}{z^{p}}\right)^{\mu}\prec 1+Az\,. \tag{2.32}$$

Let the function $g(z)$ be defined by (2.2). We then find from (2.1) and (2.32) that

$$\begin{aligned} g(z) &\prec \frac{(p-\beta)\mu}{\lambda k}z^{-\frac{(p-\beta)\mu}{\lambda k}}\int_0^z\left(1+At\right)t^{\frac{(p-\beta)\mu}{\lambda k}-1}dt \\ &= 1+\frac{(p-\beta)\mu}{\lambda k+(p-\beta)\mu}z\ . \end{aligned}$$

We now suppose that

$$\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)}f\left(z\right)}{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}=(1-\rho)h(z)+\rho\quad\left(\alpha>1;\ 0\le\rho<1;\ z\in U\right). \tag{2.33}$$

Then $h\in H[1,k]$. It follows from (2.32) and (2.33) that

$$g(z)\left\{\left(1-\lambda\right)+\lambda[(1-\rho)h(z)+\rho]\right\}\prec 1+Az\qquad\left(z\in U\right). \tag{2.34}$$

An application of Lemma 1.6 to (2.34) yields

$$\Re(h(z))>0\qquad\left(z\in U\right). \tag{2.35}$$

Combining (2.33) and (2.35), we find that

$$\Re\left(\frac{U_{0,z}^{(\alpha+1,\beta+1,\gamma+1)}f\left(z\right)}{U_{0,z}^{(\alpha,\beta,\gamma)}f\left(z\right)}\right)=(1-\rho)\Re(h(z))+\rho>\rho\qquad\left(\alpha>1;\ 0\le\rho<1;\ z\in U\right). \tag{2.36}$$

The assertion of Theorem 2.10 can now easily be derived from (1.12) and (2.36).

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Competing Interests

Authors have declared that no competing exists.

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