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On Commutative $EKFN$-ring with Ascending Chain Condition on Annihilators

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Abstract

We consider the class $\mathcal{E}$ of endo-Noetherian modules, i.e. the modules M which satisfying the ascending chain condition for endomorphic kernels: any ascending chain $Kerf_1 \subset Kerf_2 \subset \cdots \subset Kerf_n \subset \cdots$ is stationary, where $f_i \in End(M)$. Let $\mathcal{N}$ be the class of Noetherian modules. It is clear that every Noetherian R-module M is endo-Noetherian, so $\mathcal{N} \subset \mathcal{E}$, but the converse is not true. Indeed, $\mathbb{Q}$ is a non-Noetherian $\mathbb{Z}$-module which is endo-Noetherian. The aim of this work, is to characterize commutative rings for which $\mathcal{E}$ and $\mathcal{N}$ are identical.

Keywords: Annihilator; Artinian; $EKFN$-ring; endo-Noetherian; hopfian; Noetherian; strongly hopfian.

1 Introduction

The ascending chain condition is a mathematical property on orders, which used by Emmy Noether in 1921 [1] in the context of commutative algebra. The rings named, Noetherian rings, have been the subject of extensive study to modules. The descending chain condition on ideals was introduced by Artin. Rings in which any descending chain of ideals is finite are called Artinian. We say that a module is Noetherian (resp. Artinian) if any ascending (resp. descending) chain of submodules is stationary. In fact, the finiteness of the dimension of a vector space E is equivalent to the Noetherian or Artinian condition, then we have the following properties:

(i) any surjective endomorphism is injective;

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(ii) any injective endomorphism is surjective.

Condition (i) (resp. (ii)) has inspired, the concept of hopfian (resp. co-hopfian) modules. An R-module M is said hopfian if any surjective homomorphism from M to M is injective. This concept began with the introduction of hopfian groups by G. Baumslag in 1963 [2]. Since then, we focused on other issues relating to hopfian and co-hopfian properties, including:

1. the characterization of rings over which every hopfian (resp. co-hopfian, resp. fitting) module is Noetherian (resp. Artinian, resp. of finite length);
2. the characterization of rings for which every hopfian (resp. co-hopfian) module is finitely generated.

The question in 1 has been fully resolved, in the commutative case by A. Kaidi and M. Sangharé, who have established in 1988 the following results in [3]: Let R be a commutative ring, then the following conditions are equivalent:

(a) R is an I-ring;[1]

(b) R is an S-ring;[2]

(c) R is an Artinian principal ideal ring.

In 1992 M. Sangharé has established in [4]: Let R be a commutative ring, then the following conditions are equivalent:

(a) R is an I-ring;

(b) R is an S-ring;

(c) R is an F-ring;[3]

(d) R is an Artinian principal ideal ring.

The characterization of FGS-rings,[4] was fully resolved in the commutative case, by C.T. Gueye and M. Sangharé in 2004 [5]. They established that: a ring R is an FGS-ring if and only if R is an Artinian principal ideal ring.
The characterization of FGI-rings,[5] was fully resolved in the commutative case by M. Barry et all in 2005 [6]. They established that: a ring R is an FGI-ring if and only if R is an Artinian principal ideal ring. Nevertheless, the question remains open in the general case.
In 2007 A. Hmaimou et all introduced in [7], the notions of modules that involve both the conditions of chains and endomorphism, defined as follows: an R-module M is said strongly hopfian (resp. strongly co-hopfian) if for every endomorphism f of M, then $Kerf \subset Kerf^2 \subset \cdots (resp.\ Imf \supset Imf^2 \supset \cdots)$ is stationary. They showed that the class of strongly hopfian (resp. strongly co-hopfian) modules is contained the class of Noetherian (resp. Artinian) modules and the hopfian (resp. co-hopfian) modules.
Finally A. Kaidi introduced in 2009 in [8] the class of endo-Noetherian (resp. endo-Artinian) modules as follows: an R-module M is said to be endo-Noetherian (resp. endo-Artinian) if for every endomorphism $f_i \in M$, then $Kerf_1 \subset Kerf_2 \subset \cdots (resp.\ Imf_1 \supset Imf_2 \supset \cdots)$ is stationary. He showed that the class of endo-Noetherian (resp. endo-Artinian) modules is between the class of Noetherian (resp. Artinian) modules and the strongly hopfian (resp. strongly co-hopfian) modules.
The purpose of this paper is to characterize rings on which an endo-Noetherian module M is Noetherian. Such rings will be called $EKFN$-rings. We prove that a commutative ring with ascending chain condition on annihilators is an $EKFN$-ring if and only if it is an Artinian principal ideal ring.

[1] An I-ring R is a ring such that every co-hopfian R-module is Artinian.
[2] An S-ring R is a ring such that every hopfian R-module is Noetherian.
[3] An F-ring R is a ring such that every fitting R-module is of finite length.
[4] An FGS-ring R is a ring for which every hopfian R-module is finitely generated.
[5] An FGI-ring R is a ring for which every co-hopfian R-module is finitely generated.

Our paper is structured as follows: the first section covers the properties and basic aspects of certain classes of modules and rings; special attention is paid to the properties of endo-Noetherian modules. In the second section, we characterize commutative $EKFN$-rings satisfying ascending chain condition on annihilators.

2 Preliminaries

The rings considered in this section are associative with unit. Unless otherwise mentioned, all the modules considered are left unitary modules.

Proposition 2.1. *(Schur's Lemma)*
If S and T are two simple modules. Then every non-zero homomorphism is an isomorphism.

Theorem 2.1. *([9], P.220)*
For a ring R the following statements are equivalent:

1. *R is Noetherian and every prime ideal is maximal;*
2. *R is Artinian.*

Lemma 2.2. *([3], P.249)*
Let C be a local ring with maximal ideal $rC \neq 0$, where $r^2 = 0$. Let M be the total ring of fractions of the ring of polynomials $C[X]$, and σ the C-endomorphism of M defined for all $m \in M$, by $\sigma(m) = rXm$, then:

1. *$r\sigma = \sigma^2 = 0$;*
2. *If F is a C-endomorphism of M commuting with σ, then for all $m \in M$, $F(rm) = rmF(1)$.*

Proposition 2.2. *([3], P.250)*
Let R be an Artinian ring having at least one non-principal ideal. Then there exists an R-module M which is not finitely generated.

Definition 2.1. An R-module M is called endo-Noetherian if any ascending chain of endomorphic kernels $Kerf_1 \subset Kerf_2 \subset \cdots \subset Kerf_n \subset \cdots$ stabilizes, where f_i's are endomorphisms of M, i.e. there exists a positive integer n such that $Kerf_n = Kerf_{n+1}$.

Proposition 2.3. *[8]*
A ring R is endo-Noetherian if and only if R satisfies the ascending chain condition on principal left annihilators.

Proposition 2.4. *[8]*
Let M be an endo-Noetherian R-module that is a direct sum of non-zero submodules, $M = \bigoplus_{i \in I} M_i$. Then the set I is finite.

Proposition 2.5. *[8]*
Let $M_i, i \in I$, be a family of R-modules such that the set $Hom(M_i, M_j) = \{0\}$ for $i \neq j$. Let $M = \bigoplus_{i \in I} M_i$. Then M is endo-Noetherian if and only if for every $i \in I$ the module M_i is endo-Noetherian.

3 The Main Results

Let R be an associative and commutative ring with identity $1 \neq 0$.

Definition 3.1. R is said to be an $EKFN$-ring[6] if every endo-Noetherian R-module is Noetherian.

[6] $EKFN$ is derived from the two concepts namely Endo Kernel Finite and Noetherian.

Example 3.1. *A semisimple ring and an Artinian ring with principal ideal, are $EKFN$-rings.*

Proposition 3.1. *Every homomorphic image of an $EKFN$-ring is an $EKFN$-ring.*

Proof. Let R be an $EKFN$-ring and $\varphi : R \longrightarrow S$ a surjective homomorphism of rings. Let M be an S-module. Then φ induces a structure of R-module on the additive abelian group M by the following map: $R \times M \longrightarrow M$ where $(r, m) \mapsto \varphi(r) \cdot m$. Then any S-endomorphism is an R-endomorphism, and conversely.
Assume that M is an endo-Noetherian S-module. We want to show that M is Noetherian. Let $Kerf_1 \subset Kerf_2 \subset \cdots \subset Kerf_n \subset \cdots$ be an ascending chain for R-endomorphic kernels, where $f_i \in End_R(M)$. Every R-endomorphism f_i, is an S-endomorphism, thus the ascending chain for R-endomorphic kernels is an ascending chain for S-endomorphic kernels which stabilizes. So M is an endo-Noetherian R-module, then M is Noetherian. □

Proposition 3.2. *Any integral domain $EKFN$-ring is a field.*

Proof. Let R be an $EKFN$-ring. Let K be the field of fractions of R. We know that K has an R-module structure. Let f be an R-endomorphism, then $Kerf = 0$ or $Kerf = K$. It follows that any ascending chain for endomorphic kernels of K stabilizes. Then K is Noetherian. We know that the map $i : R \longrightarrow K$ is a monomorphism, so $R \subset K$. K is Noetherian, then K is a finitely generated R-module, so K is a fractional ideal of R by ([10], P.134). Thus there exists $r \in R$ such that $rK \subset R$. But $rK = K$, hence $K \subset R$. □

Corollary 3.2. *Any prime ideal of an $EKFN$-ring is maximal.*

Proof. Let P be a prime ideal of R. Let s be the canonical surjection defined by $R \longrightarrow R/P$ where $r \mapsto \tilde{r}$. By Proposition 3.1 the quotient ring of the integral domain R/P is an $EKFN$-ring, then R/P is a field. So P is maximal. □

Proposition 3.3. *Let R be an $EKFN$-ring. Then the set of all maximal ideals of R is finite.*

Proof. Let $\mathcal{L}$ be the set of all primes ideals of R. Let's show that
$Hom_R(R/P_i, R/P_j) = \{0\}$ for all $i \neq j$. Let $f \in Hom_R(R/P_i, R/P_j)$. For every prime ideal P_i, R/P_i is a simple R-module, and by Schur's Lemma f is zero or an isomorphism. Assume that f is an isomorphism, we know that $\tilde{1}_{P_j} \in R/P_j$, so there exists a unique $r \notin P_i$ such that $f(\tilde{r}_{P_i}) = \tilde{1}_{P_j}$. For all $\alpha \in P_i, f(\alpha\tilde{r}_{P_i}) = \alpha f(\tilde{r}_{P_i}) = \alpha\tilde{1}_{P_j} = \tilde{\alpha}_{P_j}$, but $f(\alpha\tilde{r}_{P_i}) = f(r\tilde{\alpha}_{P_i}) = f(\tilde{0}_{P_i}) = \tilde{0}_{P_j}$, then $\tilde{\alpha}_{P_j} = \tilde{0}_{P_j}$, so $\alpha \in P_j$. Thus $P_i \subset P_j$. Conversely for all $\alpha \in P_j, f(\alpha\tilde{r}_{P_i}) = \tilde{\alpha}_{P_j} = \tilde{0}_{P_j}$, but $f(\alpha\tilde{r}_{P_i}) = f(\widetilde{\alpha r}_{P_i}) = \tilde{0}_{P_j}$, then $\widetilde{\alpha r}_{P_i} \in P_i$, so $\alpha r \in P_i$, hence $\alpha \in P_i$. Thus $P_j \subset P_i$. Finally $P_j = P_i$, contradiction. Then $f = 0$, so $Hom_R(R/P_i, R/P_j) = \{0\}$ for all $i \neq j$. Let $M = \bigoplus_{i \in I} R/P_i$, let's show that M is endo-Noetherian. We know that R/P_i is endo-Noetherian. By Proposition 2.5, M is endo-Noetherian. Now it follows from Proposition 2.4 that I is finite. □

Proposition 3.4. *Let C be a local ring with maximal ideal $rC \neq 0$, where $r^2 = 0$. Let M be the total ring of fractions of the ring of polynomials $C[X]$, and σ the C-endomorphism of M defined for all $m \in M$, by $\sigma(m) = rXm$. Then every non-zero C-endomorphism F of M commuting with σ is such that either $KerF = rM$, or is a monomorphism.*

Proof. Note that a ring element $m \in M$ is invertible if and only if $m \notin rM$. Indeed, let $m \in rM$, and assume that m is invertible in M i.e. there exists $m' \in M$ such that $mm' = m'm = 1$. Since $m = rm_1$ where $m_1 \in M$, we have $rm_1m' = 1$, then $r^2m_1m' = r = 0$. Contradiction as $r \neq 0$. Conversely, if $m \notin rM$, then $m \notin rC$. As rC is a maximal ideal of the local ring C, then m is invertible in C, hence in M.

If $F(1)$ is not invertible, then $F(1) \in rM$. For any $m \in rM$ there exists $q \in M$ such that $m = rq$. Hence by Lemma 2.2, $F(m) = F(rq) = rqF(1) = mF(1) = 0$. So $rM \subset KerF$ $(*)$. Let's show that in this case for all $m \notin rM$, then $F(m) \neq 0$. We have for all $m \in M \setminus rM$, $rm \in rM \setminus \{0\}$, then $F(rm) = rmF(1) = 0$. Or $F(rm) = rF(m) = 0$, so $F(m) \in rM$ i.e. there exists $\alpha \in M$ such that $F(m) = r\alpha$. Assume that $\alpha \in rM$, so $F(m) = 0$. Then $m \in KerF$, i.e. $M \setminus rM \subset KerF$ $(**)$. Hence $(*)$ and $(**)$ imply that $M \subset KerF$, then F is identically zero (impossible by assumption). Hence, it follows that $\alpha \notin rM$. So $F(m) = r\alpha \neq 0$. Then, for all $m \notin rM$ $F(m) \neq 0$. Hence $m \in KerF \Rightarrow m \in rM$ so $KerF \subset rM$ $(***)$. Thus $(*)$ and $(***)$ imply that $KerF = rM$.
If $F(1)$ is invertible, i.e. $F(1) \notin rM$. Let m be a non-zero element of M. Let's show that $F(m) \neq 0$. In case $m \in rM$, then $F(m) = mF(1)$. Hence $F(m) \neq 0$. If now $m \notin rM$, then $rm \neq 0$, so $F(rm) = rmF(1)$, hence $F(rm) = rF(m) = rmF(1) \neq 0$. Then $F(m) \neq 0$. Thus F is a monomorphism. □

Theorem 3.3. ***Characterization theorem***
Let R be a ring satisfying acc on annihilators, the following conditions are equivalent:

1. *R is an $EKFN$-ring;*
2. *R is an Artinian principal ideal ring.*

Proof.
- Assume that R is an $EKFN$-ring. Since R satisfies the acc on annihilators, then by Proposition 2.3, R is endo-Noetherian, so R is Noetherian. We know by Corollary 3.2 that any prime ideal of an $EKFN$-ring is maximal. Thus R is Artinian. Assume that R has at least one non-principal ideal, so there exists an R-module M which is not finitely generated by Proposition 2.2. So M is non-Noetherian. Let $Kerf_1 \subset Kerf_2 \subset \cdots \subset Kerf_n \subset \cdots$ be an ascending chain for R-endomorphic kernels, where $f_i \in End_R(M)$. Note that R-endomorphisms of M are C-endomorphisms of M which commute with σ defined in Lemma 2.2. Hence any ascending chain for R-endomorphic kernels stabilizes. So M is endo-Noetherian, contradiction. Thus R is an Artinian principal ideal ring.
- Assume that R is an Artinian principal ideal ring. Then, R is an $EKFN$-ring.

□

4 Conclusion

Let R be an associative and commutative ring with identity $1 \neq 0$. Every Noetherian R-module M is endo-Noetherian, but the converse is not true. R is said to be an $EKFN$-ring if every endo-Noetherian R-module is Noetherian. We have studied and characterized partially in this paper $EKFN$-ring. Authors think that a characterization of $EKFN$-ring in nonsingular ring can be very interesting.

Competing Interests

The authors declare that no competing interests exist.

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