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A Mathematical Model to Predict the Prevalence and Transmission Dynamics of Tuberculosis in Amansie West District, Ghana

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Abstract

In this paper, a Susceptible - Exposed - Infected - Recovered (SEIR) epidemiological model is formulated to determine the transmission of tuberculosis. The equilibrium points of the model are found and their stability is investigated. By analyzing the model, a threshold parameter R_0 was found which is the basic reproductive number. It is noted that when $R_0 < 1$ the disease will fail to spread and when $R_0 > 1$ the disease will persist in the population and become endemic. The model has two non–negative equilibria namely the disease – free equilibrium and the endemic equilibrium. The graphical solutions of the differential equations were developed using Matlab as well as the computer simulations.

Keywords: Differential equations, exposed and infected, simulation, transmission dynamics, tuberculosis.

1 Introduction

Tuberculosis, MTB, or TB (short for tubercle bacillus) is a common, and in many cases lethal, infectious disease caused by various strains of mycobacteria, usually Mycobacterium tuberculosis [1]. Tuberculosis typically attacks the lungs, but can also affect other parts of the body. It is spread through the air when people who have an active TB infection cough, sneeze, or otherwise transmit their saliva through the air [2]. Most infections are asymptomatic and latent, but about one in ten latent infections eventually progresses to active disease which, if left untreated, kills more than 50% of those infected. Tuberculosis (TB) describes an infectious disease that has plagued humans since the Neolithic times [3].

Physicians in ancient Greece called this illness phthisis to reflect its wasting character. During the 17th and 18th centuries, TB caused up to 25% of all deaths in Europe [4].

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When people with active pulmonary TB cough, sneeze, speak, sing, or spit, they expel infectious aerosol droplets 0.5 to 5.0 μm in diameter. A single sneeze can release up to 40,000 droplets [5]. Each one of these droplets may transmit the disease, since the infectious dose of tuberculosis is very low, the inhalation of fewer than 10 bacteria may cause an infection [6].

A number of factors make people more susceptible to TB infections. The most important risk factor globally is HIV; 13% of all TB cases are infected by the virus [7]. This is a particular problem in sub-Saharan Africa, where rates of HIV are high [8] and [12].

Tuberculosis is closely linked to both overcrowding and malnutrition, making it one of the principal diseases of poverty [9]. Those at high risk thus include: people who inject illicit drugs, inhabitants and employees of locales where vulnerable people gather, medically underprivileged and resource-poor communities, high-risk ethnic minorities, children in close contact with high-risk category patients and health care providers serving these clients [10].

Tuberculosis in Ghana started way back in the pre-independence era when the then colonial government recognized the need to combat the disease due to the threat it posed to the larger society. In July 1954, the Ghana Society for the Prevention of Tuberculosis was established to support and supplement government's efforts at combating the disease [11].

Over 46,000 new cases of TB annually are estimated by the World Health Organization (WHO) in Ghana [12]. However, in reality, less than a third of the estimated number of cases is officially reported each year by health facilities in the country. [13] sought to reduce the infectious group by the reduction of the contact between infectious and exposed individuals.

The first mathematical model of TB was presented by [14]. Following this, there were several numerical studies, primarily focusing on cost-effectiveness of different interventions [15,16]. [16] used a model with one progression rate and various latent classes representing different treatment and control strategies, and argued that vaccination was cost-effective in countries with high TB burdens. Waaler continued his work in 1968 and 1970, [17] and [18], little work on models of tuberculosis appeared in the literature until the mid - 1990's.

The Amansie West District is one of the 27 districts in the Ashanti region of Ghana. It was carved out of the Amansie East District of the Ashanti Region in 1989, as part of the Government's Decentralization Policy. Manso-Nkwanta is the district capital. The district is located in the south-western part of the Ashanti Region. It shares common boundary on its Western part with the Atwima Mponua District. On its northern part can be found the Atwima Nwabiagya and the Bosomtwe – Atwima - Kwanhuma Districts, while a regional boundary separates it from Western and Central Region on its southern part. The predominant occupation is subsistence farming and illegal small scale mining popularly known as Galamsey. Due to the activities of the illegal mining, the district is faced with the problem of immigration making the district exposed to infectious diseases with tuberculosis being prevalent.

2. Materials and Methods

The mathematical model will be formulated using differential equations based on the epidemiological compartment modelling. The computer software package that will be used to solve the differential equation model numerically is Matrix Laboratory (Matlab R2010a). Analysis

and numerical simulations of the model will be conducted. Records showing the trend of tuberculosis cases recorded annually were obtained from the Amansie West District Health Directorate, from 2002 - 2011.

In view of the above, the main gap in knowledge to be filled by the paper was to find out the mode of transmission of Tuberculosis and using the SEIR model based on data obtained to analyse it to see whether the disease will be endemic or not when there is an outbreak in the district.

2.1 The Model

We consider the standard SEIR model where the individuals in the population are divided into four compartments (Fig. 1). The susceptibles (*S*) which refers to the healthy individuals that has not yet come into contact with TB bacterium. The exposed (*E*) are people who have come into contact with the disease but are not yet infective or infectious. The infective *(I*) are those who have become infected with TB and are able to transmit the disease and the recovered (*R*) are individuals who have recovered from TB. The proportions of the individuals in the compartment of the population, i.e. S, E, I and R, at time t is denoted as S(t), E(t), I(t) and R(t) respectively.

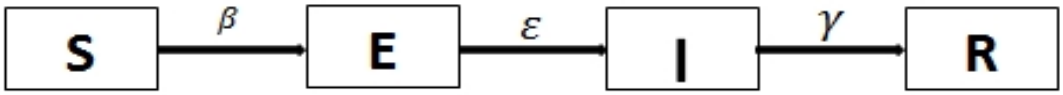

Fig. 1. Flow chart showing the SEIR model

2.2 Model Assumptions

1. Age, sex, social status, race coupled with climatic conditions in the district does not affect the probability of an individual being infected.
2. The death rate of all individuals is balanced by a birth rate (birth and deaths occurs at equal rates).
3. We assume that an individual is firstly exposed to the disease through interaction with the infectious but does not become infectious instantly.
4. The disease in transmitted in a closed environment. There is no emigration or immigration and there is neither birth nor death in the population. Hence the total population, N of individuals in the district remains constant.
5. Those that are recovered become immuned and are educated about the transmission of the disease. The transmission of the disease within the sanatorium is neglected.
6. Lastly, we assume that there is no treatment failure in the sanatorium and therefore a patient will recover or die.

2.3 Model Equations

Applying the assumption here, the total population, N is at full capacity (it value is between 0 and N).

Also, the individuals are likely to be infected by the infectious individuals in case of contact except those who are immune. Those who are undetected or late detected infectious are the ones

contributing to the disease transmission and spread. Those detected are isolated to the hospital for immediate treatment and education.

The birth and death rate is μ . Hence μN will be the rate at which individuals are born into the susceptible class without any immunity and μS is the rate at which the leave it through death.

The rate at which the susceptible class changes is equal to the rate at which infections occur. This occurs when the disease is passed from an infective individual to a susceptible one. The number of susceptible – infective contacts is proportional to the product of $S(t)$ and $I(t)$.

Hence the rate of change in the susceptible individuals is

$$\frac{dS}{dt} = \mu N - \mu S - \frac{\beta SI}{N}$$, where βSI is rate of infection.

The rate at which individuals leave the susceptible population is equal to the rate at which they enter the exposed population.

The number of individuals in the exposed class increases since those in the susceptible class becomes exposed to the disease.

Let ε the rate which an exposed individual becomes infectious. Then the rate of change of the exposed population is given by

$$\frac{dE}{dt} = \frac{\beta SI}{N} - (\mu + \varepsilon)E$$

Let γ be the rate at which an infected individual may recover. The rate of change of the infective individuals will be

$$\frac{dI}{dt} = \varepsilon E - (\mu + \gamma)I$$

The rate of change of the recovered is given by

$$\frac{dR}{dt} = \gamma I - \mu R$$

This leads to the following formulations of the SEIR model from the description, assumptions and compartmental diagram and was given as follows:

$$\frac{dS}{dt} = \mu N - \mu S - \frac{\beta SI}{N} \quad (1)$$

$$\frac{dE}{dt} = \frac{\beta SI}{N} - (\mu + \varepsilon)E \tag{2}$$

$$\frac{dI}{dt} = \varepsilon E - (\mu + \gamma)I \tag{3}$$

$$\frac{dR}{dt} = \gamma I - \mu R \tag{4}$$

The nonlinear system of differential equations formulated above has initial conditions $S(0) = S_0 > 0, E(0) = E_0 > 0, I(0) = I_0 > 0$ and $R(0) = R_0 > 0$.

Also, the rate of contact, the rate of infection, the recovery rate and the birth and death rate are all non-negative $(\beta > 0, \varepsilon > 0, \gamma > 0 and\, \mu > 0$ respectively).

Hence $N(t) = S(t) + E + I(t) + R(t)$ and

$$\frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0 \tag{5}$$

Since $\frac{dN}{dt} = 0$ and thus $N = S + E + I + R$ is a constant.

$$\text{If we let} \quad \left.\frac{S(T)}{N} = s(t), \frac{E(T)}{N} = e(t), \frac{I(t)}{N} = i(t), and, \frac{R(t)}{N} = r(t)\right\} \tag{6}$$

Where $s(t)$, $e(t)$, $i(t)$ and $r(t)$ are the respective fractions of the population for susceptible, exposed, infective and recovered individuals (variables and definitions in Table 1).

Then $s(t) + e(t) + i(t) + r(t) = 1$

Table 1. Variables and definitions

Variable	Definition
S(t)	The number of Susceptible individuals in the population at time, t
E(t)	The number of Exposed individuals in the population at time, t
I(t)	The number of Infected individuals in the population at time, t
R(t)	The number of Recovered individuals in the population at time, t
N	Total population

Table 2. Parameters and their definitions

Parameter	Definition
β	The rate of contact. It is defined as the average number of effective contacts with other individuals (susceptible) per infective unit time.
ε	The rate at which the exposed individuals become infective or infectious.
γ	The rate at which the infectious individuals recover per unit time.
μ	The birth and death rate

The SEIR model is similar to the SIR endemic model except that it has an additional compartment E or the exposed class. The parameters and their definitions can be found in Table 2. This compartment reflects an inclusion of incubation period. Individuals in this class are infected but not infectious.

We divide equations (1) and (2) by N and then substitute equation (6) into equations (1) – (4) respectively to obtain

$$\frac{ds}{dt} = \mu - (\mu + \beta i)s \tag{7}$$

$$\frac{de}{dt} = \beta si - (\mu + \varepsilon)e \tag{8}$$

$$\frac{di}{dt} = \varepsilon e - (\mu + \gamma)i \tag{9}$$

$$\frac{dr}{dt} = \gamma i - \mu r \tag{10}$$

With initial conditions $s(0) = s_0 \geq 0, e(0) = e_0 \geq 0, i(0) = i_0 \geq 0, r(0) = r_0 \geq 0$

2.4 Basic Reproductive Number, R_0

In epidemiology, the basic reproductive number (sometimes basic reproduction rate or ratio) of an infection is the number of cases one case generates on the average over the course of its infectious period.

The basic reproduction number R_0 is the threshold for many epidemiological models. When $R_0 < 1$, the infection dies out in the long run (i.e. each infected individual produces one average less than one new infected individual). One the other hand, if $R_0 > 1$, the infection will be able to

spread in a population (i.e. each infected individual produces more than one new infected individual).

For the purpose of our model, we use the approach by [19] to determine the basic reproductive number for this thesis and is given by

$$R_0 = \frac{\beta\varepsilon}{(\mu+\varepsilon)(\mu+\gamma)} \tag{11}$$

Also, the number of contacts between susceptible and infective is given by

$$\sigma = \frac{\beta}{\gamma} \tag{12}$$

The SEIR model always has at least one solution given by $s = 1$, and $e = i = r = 0$. If the threshold quantity is greater than one then there is a unique endemic equilibrium solution in D where $D = \left\{(e,i,r) \middle| e \geq 0, i \geq 0, r \geq 0, e+i+r \leq 1\right\}$, [20].

2.5 The Herd Immunity Threshold, H_1

The herd immunity threshold (H_1) is the percentage of the population that needs to be immunized to control the transmission of the disease. The endemicity of a disease depends on the basic reproduction number, R_0. This threshold value can predict whether the disease will approach and spread through the population or not.

The equation given by [21] for estimating the herd immunity threshold is given as

$$H_1 = 1 - \frac{1}{R_0} \tag{13}$$

2.6 Analysis of the Model

In this section, we present the results of stability analysis of the equilibrium points.

2.7 Equilibrium of the Model

The system of equations of the model (Equations 7 - 10) has two non-negative equilibrium points, namely, the disease-free equilibrium, where $i = 0$ and endemic equilibrium, where $i \neq 0$. The equilibrium point for the disease – free equilibrium is given as:

$$\left(s^*, e^*, i^*\right) = (1,0,0) \tag{14}$$

The characteristic equation about the point (1,0,0) is as follows:

$$(\mu+\lambda)\left[\lambda^2+(2\mu+\varepsilon+\gamma)\lambda+(\mu+\varepsilon)(\mu+\gamma)(1-R_0)\right]=0$$

where the roots are:

$$\lambda_1=-\mu,$$

$$\lambda_2=\frac{-(2\mu+\varepsilon+\gamma)-\sqrt{\left((2\mu+\varepsilon+\gamma)^2-4(\mu+\varepsilon)(\mu+\gamma)(1-R_0)\right)}}{2},$$

$$\lambda_3=\frac{-(2\mu+\varepsilon+\gamma)+\sqrt{\left((2\mu+\varepsilon+\gamma)^2-4(\mu+\varepsilon)(\mu+\gamma)(1-R_0)\right)}}{2}.$$

It is clear that λ_1 and λ_2 are negative. Moreover, λ_3 is negative when $R_0<1$ and it is positive when $R_0>1$. Hence, we deduce that:

- If $R_0<1$, then the disease-free equilibrium, (1,0,0), is locally asymptotically stable.
- If $R_0>1$, then (1,0,0) is unstable.

The endemic equilibrium points for the respective states are as follows:

$$(s_e,e_e,i_e,)=\left(\frac{1}{R_0},\frac{\mu(R_0-1)}{(\mu+\varepsilon)R_0},\frac{\mu(R_0-1)}{\beta}\right) \tag{15}$$

2.8 Stability of the Equilibria

To determine the stability of the system, we will consider linearizing the systems of equations (7) – (9) about the equilibrium points by taking the Jacobian of them.

The Jacobian matrix is given by

$$J(s,e,i)=\begin{bmatrix}-\mu-\beta i & 0 & -\beta s\\ \beta i & -\mu-\varepsilon & \beta s\\ 0 & \varepsilon & -\mu-\gamma\end{bmatrix} \tag{16}$$

For the disease-free equilibrium, we evaluate the Jacobian matrix at the equilibrium points $(s^*,e^*,i^*)=(1.0.0)$ and hence we get

$$J(s^*, e^*, i^*) = \begin{bmatrix} -\mu & 0 & -\beta \\ 0 & -\mu - \varepsilon & \beta \\ 0 & \varepsilon & -\mu - \gamma \end{bmatrix} \tag{17}$$

Solving for roots of the characteristic polynomial (with real coefficient) given in the Jacobian matrix leads to the characteristic equation given as:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{18}$$

Where

$$\left.\begin{aligned} a_1 &= (3\mu + \varepsilon + \gamma) \\ a_2 &= \left[(\mu + \varepsilon)(\mu + \gamma) - \beta\varepsilon + \mu(2\mu + \varepsilon + \gamma)\right] \\ a_3 &= \mu\left[(\mu + \varepsilon)(\mu + \gamma) - \beta\varepsilon\right] \end{aligned}\right\} \tag{19}$$

Using the Routh – Hurwitz stability criterion analysis we talked about earlier, the conditions $a_1 > 0$, $a_3 > 0$ and $a_1 a_2 - a_3 > 0$ must hold. If these conditions are true, then all roots of the characteristic polynomial equation have negative real part, which concludes that there is a stable equilibrium.

The endemic steady state has equilibrium point given by

$$(s_e, e_e, i_e,) = \left(\frac{1}{R_0}, \frac{\mu(R_0 - 1)}{(\mu + \varepsilon)R_0}, \frac{\mu(R_0 - 1)}{\beta}\right)$$

The Jacobian matrix of this equilibrium point is evaluated to obtain

$$J(s_e, e_e, i_e) = \begin{bmatrix} -\mu R_0 & 0 & \dfrac{-(\mu + \varepsilon)(\mu + \gamma)}{\varepsilon} \\ \mu(R_0 - 1) & -(\mu + \varepsilon) & \dfrac{(\mu + \varepsilon)(\mu + \gamma)}{\varepsilon} \\ 0 & \varepsilon & -(\mu + \gamma) \end{bmatrix} \tag{20}$$

Solving for the roots of the polynomial in the Jacobian matrix leads to the characteristic equation

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0 \tag{21}$$

Where

$$\left.\begin{aligned} b_1 &= \varepsilon + \gamma + \mu\left(2 + R_0\right) \\ b_2 &= \mu R_0 (2\mu + \varepsilon + \gamma) \\ b_3 &= \mu\left(R_0 - 1\right)\left[\mu^2 + \mu(\varepsilon + \gamma) + \varepsilon\gamma\right] \end{aligned}\right\} \tag{22}$$

From the Routh – Hurwitz stability criterion, we need to determine the stability of the characteristic equation above. From the theorem, if the conditions $b_1 > 0$, $b_3 > 0$ and $b_1 b_2 - b_3 > 0$ are true, then all the roots of the characteristic equation have negative real part, which means a stable equilibrium.

From equation (22), the conditions are true for $R_0 > 1$ since $b_1 > 0$, $b_3 > 0$ and $b_1 b_2 - b_3 > 0$ (all non-negative), for all the values of the parameter and $R_0 > 1$, hence it is also true. In conclusion, by the Routh – Hurwitz stability criteria, the endemic steady state is stable when $R_0 > 1$.

Also, the endemic steady state is locally asymptotically stable when $R_0 > 1$, because $b_1 > 1,\ b_3 > 1$ and $b_1 b_2 - b_3 = \mu R_0 \left(\varepsilon + \mu\right)^2 + \left(\gamma + \mu + \mu R_0\right) b_2 + \left(\mu + \varepsilon\right)\left(\mu + \gamma\right) > 0.$

3. Numerical Analysis and Results

Records obtained from the Amansie West district Health Directorate shows that tuberculosis is an endemic in the district. See Appendix II for the data. For instances, from 2007 to 2011, there has been an increase in the number of cases recorded. In 2007, 34 cases were diagnosed, 2008 (62 cases), 2009 (59 cases), 2010 (62 cases) and 2011 (70 cases). Fig. 2 shows the trend of tuberculosis cases recorded in the district from 2002 – 2011.

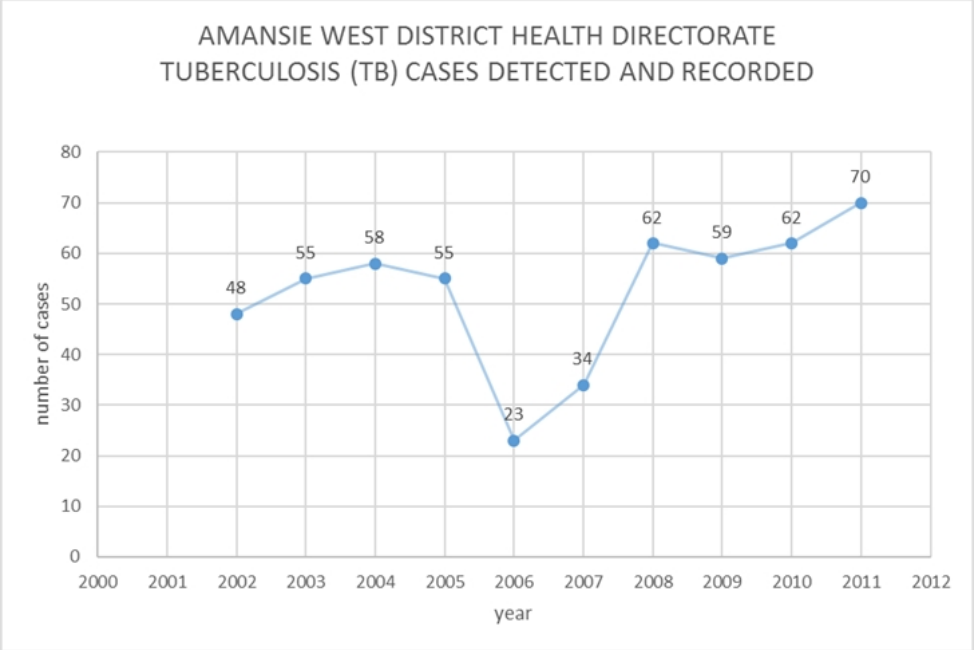

Fig. 2. Trend of Tuberculosis cases in the Amansie West District from 2002 – 2011

The table below shows the estimates of the parameters used in the model.

Table 3. Parameters and their estimated values used in the model

Parameter	Symbol	Value
Death rate	μ	0.00875
Transmission rate	β	0.5853
Infectious rate	ε	0.1666
Recovery rate	γ	0.5000

The values of the parameter estimates from Table 3 are substituted into equations (7) – (10) to obtain

$$\frac{ds}{dt} = 0.00875 - (0.00875 + 0.5853i)s \tag{23}$$

$$\frac{de}{dt} = 05853i - 0.17535e \tag{24}$$

$$\frac{di}{dt} = 0.1666e - 0.50875i \tag{25}$$

$$\frac{dr}{dt} = 05i - 0.00875r \tag{26}$$

3.1 Estimation of the Basic Reproductive Number, R_0

From equation (11), the basic reproduction number for the SEIR model is given by

$$R_0 = \frac{\beta\varepsilon}{(\mu+\varepsilon)(\mu+\gamma)} = \frac{0.5853 \times 0.1666}{(0.00875+0.1666)(0.00875+0.5)} = 1.09305 > 1$$

Since $R_0 > 1$, the prevalence of Tuberculosis will result in an epidemic. This is due to the fact that the rate of transmission is greater than the recovery rate.

The number of contacts between susceptible individuals and the infective ones is calculated from equation (12) and is given by

$$\sigma = \frac{\beta}{\gamma} = \frac{0.5853}{0.5000} = 1.1706$$

This shows that an average of one tuberculosis patient contacts 1.1706 susceptible individuals during an infectious period.

3.2 Estimation of the Herd Immunity Threshold, H_1

From equation (13), the herd immunity threshold is given as

$$H_1 = 1 - \frac{1}{1.09305} = 0.0851$$

Therefore, to control an epidemic, about 8.51% of the population has to be immunized when there is an outbreak.

3.3 Estimation of the Equilibrium Points

From equation (14), the equilibrium point of the disease – free steady state was determined to be

$$(s^*, e^*, i^*) = (1, 0, 0)$$

The estimates of the endemic steady state equilibrium were determined from equation (15) and is given by

$$\begin{aligned}\left(s_e, e_e, i_e\right) &= \left(\frac{1}{R_0}, \frac{\mu(R_0-1)}{(\mu+\varepsilon)R_0}, \frac{\mu(R_0-1)}{\beta}\right) \\ &= \left(\frac{1}{1.09305}, \frac{0.00875(1.09305-1)}{(0.00875+0.1666)\times 1.09305}, \frac{0.00875(1.09305-1)}{0.5853}\right) \\ &= (0.914871, 0.00042479, 0.00139106)\end{aligned} \tag{27}$$

3.4 Stability Analysis of the Equilibrium Points

The disease – free equilibrium point for the model was determined as $\left(s^*, e^*, i^*\right) = (1,0,0)$. The stability of the equilibrium point using the Jacobian matrix from equation (17) was determined. Hence, we obtain

$$J\left(s^*, e^*, i^*\right) = \begin{bmatrix} -0.00875 & 0 & -0.5853 \\ 0 & -0.17535 & 0.5853 \\ 0 & 0.1666 & -0.50875 \end{bmatrix}$$

The determination of the roots of the characteristic polynomial equation from the equation (19), is given in the Jacobian matrix as follows:

$$a_1 = \left[(3\times 0.00875) + 0.1666 + 0.5\right] = 0.69285$$

$$a_2 = \left[(0.00875+0.1666)(0.00875+0.5) - (0.5853\times 0.1666) + 0.00875\left((2\times 0.00875)+0.1666+0.5\right)\right] = -2.3158\times 10^{-3}$$

$$a_3 = 0.00875\left[(0.00875+0.1666)(0.00875+0.5) - (0.5853\times 0.1666)\right] = -7.2639\times 10^{-5}$$

Note:
$$a_1 a_2 - a_3 = \left[\left(0.69285\times -2.3158\times 10^{-3}\right) - \left(-7.2639\times 10^{-5}\right)\right] = -1.5318\times 10^{-3}$$

Therefore, the characteristic equation is given by

$$\lambda^3 + 0.69285\lambda^2 - 2.3158\times 10^{-3}\lambda - 7.2639\times 10^{-5} = 0 \tag{28}$$

The results indicates that $a_1 > 0$, $a_2 < 0$, and $a_1 a_2 - a_3 < 0$. From the Routh – Hurwitz stability criterion, these conditions do not hold. Hence, the disease – free equilibrium is an unstable steady state. This means that when an individual infected with mycobacterium

tuberculosis is present in a susceptible population, it will eventually result in an outbreak of the disease.

From equation (27), the endemic equilibrium point was given as

$(s_e, e_e, i_e) = (0.914871, 0.00042479, 0.00139106)$. The Jacobian matrix corresponding to the endemic equilibrium point from equation (20) gives

$$J(s_e, e_e, i_e) = \begin{bmatrix} 9.56141875\times10^{-3} & 0 & -0.53547 \\ 8.141875\times10^{-4} & -0.17535 & 0.53547 \\ 0 & 0.1666 & -0.50875 \end{bmatrix}$$

The roots of the characteristic polynomial equation given in equation (21) are obtained as follows

$$b_1 = [0.1666 + 0.5 + 0.00875(2 + 1.09305)] = 0.69366$$
$$b_2 = [0.00875 \times 1.09305(2 \times 0.00875 + 0.1666 + 0.5)] = 6.5429 \times 10^{-3}$$
$$b_3 = 0.00875(1.09305 - 1)[0.00875^2 + 0.00875(0.1666 + 0.5) + (0.1666 \times 0.5)] = 7.2633 \times 10^{-5}$$

Note: $b_1 b_2 - b_3 = 0.69366 \times (6.5429 \times 10^{-3}) - (7.2633 \times 10^{-5}) = 4.4660 \times 10^{-3}$

The characteristic equation is given by

$$\lambda^3 + 0.69366\lambda^2 + 6.5429 \times 10^{-3}\lambda + 7.2633 \times 10^{-5} = 0 \quad (29)$$

The values of the roots of the characteristic polynomial equation satisfy the Routh – Hurwitz stability criterion. This is because $b_1 > 0$, $b_2 > 0$, and $b_1 b_2 - b_3 > 0$ .Therefore, the endemic steady state is asymptotically stable.

3.5 Sensitivity Analysis

Sensitivity analysis deals with the study of how the uncertainty in the output of a mathematical model or system (numerical or otherwise) can be apportioned to different sources of uncertainty in its inputs [22].

The effect on the reproduction number, R_0 and the stability of the disease – free equilibrium was analysed. This will be done when the value of the parameter β changes whilst μ, ε and γ remain the same and also, when γ changes whilst μ, β and ε are maintained.

1. If $\beta$is increased and γ is maintained, i.e. $\beta = 0.6853$ and $\gamma = 0.5$,

$$\left.\begin{aligned} &\Rightarrow R_0 = 1.2798 > 1 \\ &a_1 = 0.69285 > 0 \\ &a_2 = -0.018975 < 0 \\ &a_3 = -0.000218 < 0 \\ &a_1 a_2 - a_3 = -0.012928 < 0 \end{aligned}\right\}$$

From these results, the disease-free equilibrium is unstable.

2. If β is reduced and γ is maintained. i.e. $\beta = 0.4853$ and $\gamma = 0.5$,

$$\left.\begin{aligned} &\Rightarrow R_0 = 0.9063 < 1 \\ &a_1 = 0.69285 > 0 \\ &a_2 = 0.014344 > 0 \\ &a_3 = 0.000731 > 0 \\ &a_1 a_2 - a_3 = 0.009207 > 0 \end{aligned}\right\}$$

Hence, a stable disease – free equilibrium is attained.

From the above results, as the transmission rate increases or the recovery rate decreases, $R_0 > 1$ and the disease – free equilibrium is unstable. This indicates that, the disease will spread when there is an outbreak. Apparently, as the transmission rate decreases or the recovery rate increases, $R_0 < 1$ and hence the disease – free equilibrium will be stable. This therefore means that the disease will not spread.

The same scenarios as we did for the disease – free equilibrium.

1. If β increases and γ remains the same. i.e. $\beta = 0.6853$ and $\gamma = 0.5$,

$$\left.\begin{aligned} &\Rightarrow R_0 = 1.2798 > 1 \\ &b_1 = 0.69529 > 0 \\ &b_2 = 0.00766 > 0 \\ &b_3 = 0.000218 > 0 \\ &b_1 b_2 - b_3 = 0.005108 > 0 \end{aligned}\right\}$$

The endemic equilibrium is stable.

2. If the β is reduced and γ remains the same. i.e. $\beta = 0.4853$ and $\gamma = 0.5$,

$$\left.\begin{aligned} &\Rightarrow R_0 = 0.9063 < 1 \\ &b_1 = 0.69203 > 0 \\ &b_2 = 0.005424 > 0 \\ &b_3 = -0.0000755 < 0 \\ &b_1 b_2 - b_3 = 0.00382 > 0 \end{aligned}\right\}$$

Hence, the endemic equilibrium is unstable.

When the transmission rate increases or the recovery rate decreases, $R_0 > 1$ and hence the endemic equilibrium is stable. This means that the disease will spread when there is an outbreak. On the other hand, as the transmission rate decreases or the recovery rate increases, $R_0 < 1$, and the endemic equilibrium is in an unstable state. This therefore means that the disease will not spread.

3.6 Sensitivity Analysis by Simulation

Numerical simulations on the SEIR model for tuberculosis using the data were done. Matlab R2010a was used and the value of the parameters are found in Table 3. The Matlab codes are found in the appendix. The effects and the changes that will occur in the model when the values of each of the compartments of the model were altered .i.e. Susceptible(S), Exposed(E), Infected(I) and the Removed(R) were looked at. Time was measured in months for a period of one year depending on the period of how tuberculosis prevalence occurs. Some assumptions and graphing to see the effect of changes in each compartment of the model.

A total population of 500 individuals in which all belonged to the susceptible was started with. This implies that there were no exposed, infected and recovered individuals in the population. The simulation gave the following graphs as shown in Fig. 3.

It can be noticed from the graph that, the number of susceptibles was 500 at the period of time under study whiles the number of exposed, infected and recovered are all at a constant number of zero during the entire period under study. This implies that there will not be any effect on each compartment when all the population belong to the susceptible.

Further adjustment of the system was analysed to obtain the results by introducing 10 infected, thereby reducing the susceptibles to 490, no exposed and no recovered individuals into the system.

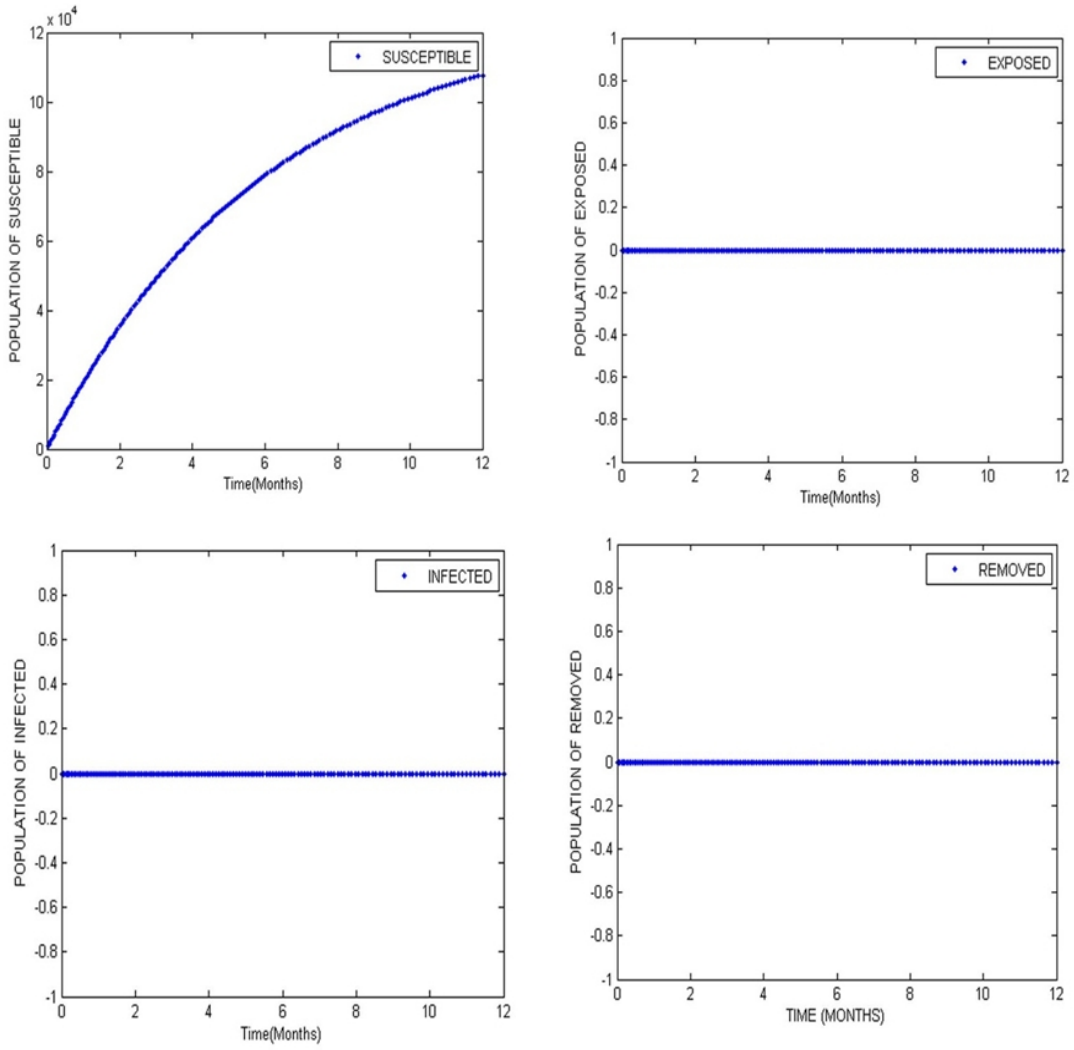

Fig. 3. S = 500, E = 0, I = 0, R = 0

It will be observed from Fig. 4 that the number of susceptibles increased with time and became constant getting to the end of the period under study. There is also, no effect on the exposed individuals and the 10 infected reduced with time to zero. The 10 infected moves into the recovered compartment and no infection are recorded again. This may be due to early detection and probably seeking immunization. This will eventually cause the disease to die the anyway. By so doing, the recovered individuals begin to increase as the infected recovered and with time, they all move back to the susceptible compartment.

The final part of the simulation place emphasis on the exposed class. This will help put the disease under control in the district. With the introduction of some exposed individuals into our system, the number of infected tend to decrease but with time, they increase later. The susceptible class

was stable at a point in time within the period of study but increase in number afterwards. When we introduce the infected into our system, with no exposed individuals, the disease failed to spread within some short period of time. The introduction of some exposed individuals will increase the infected after some period of time and hence the disease will spread again. All can be found be in Fig. 5.

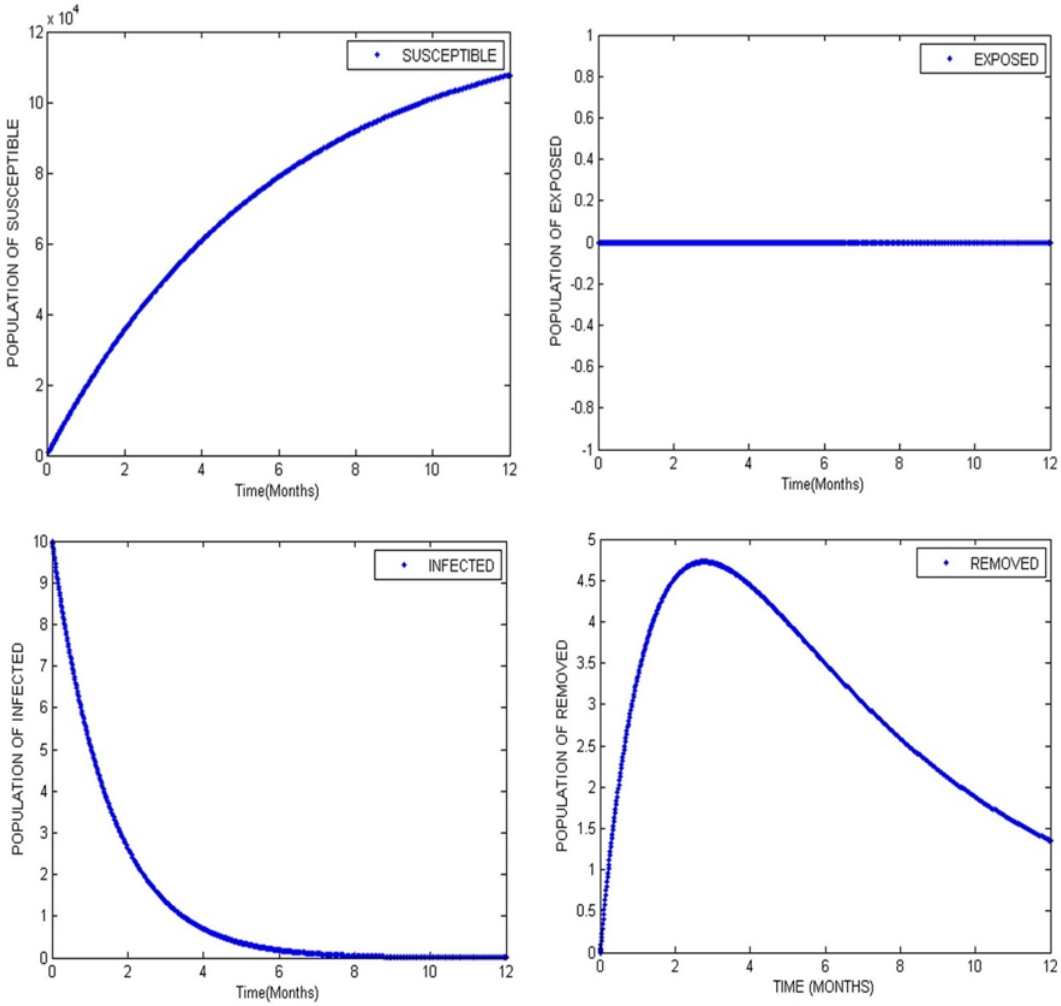

Fig. 4. S = 490, E = 0, I = 0, R = 0

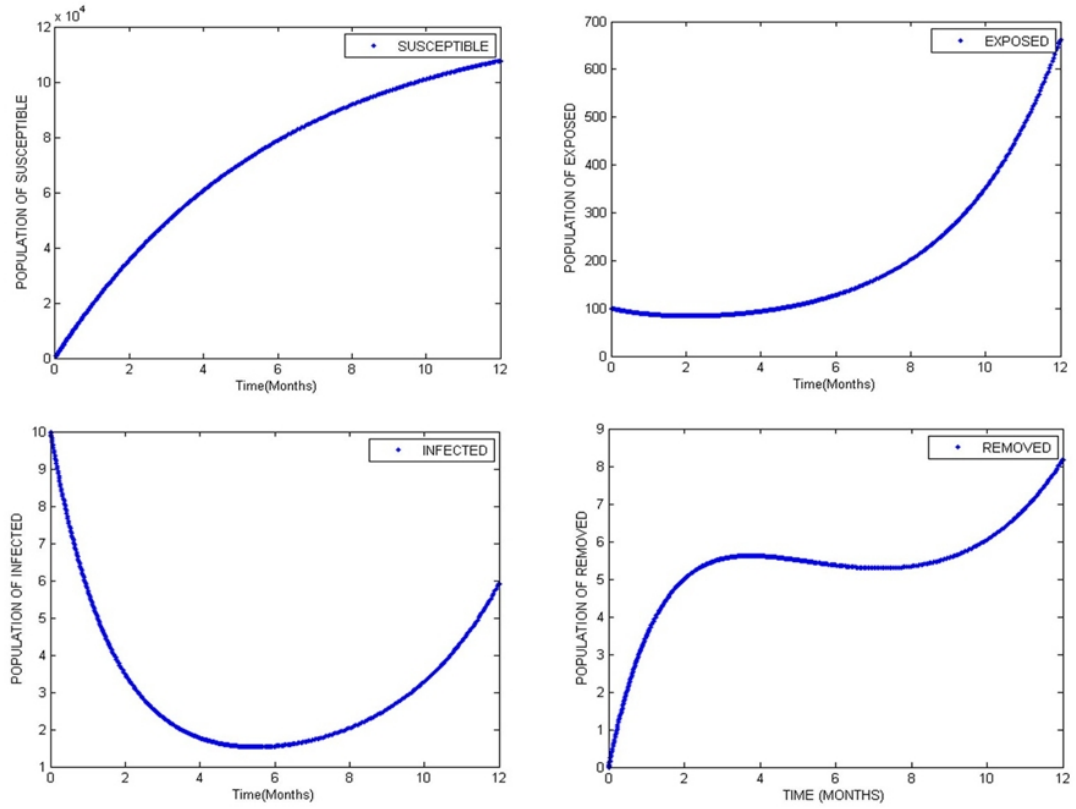

Fig. 5. S = 390, E = 100, I = 10, R = 0

4. Discussion of Results

In this paper, a mathematical model of the prevalence and transmission of tuberculosis in Amansie West district of the Ashanti Region was studied. The standard SEIR differential equation model was used to predict the transmission and spread of tuberculosis. By analyzing the model, we found a threshold parameter, R_0, was found which is the basic reproductive number. It was noted that when $R_0 < 1$, then the epidemic will not spread and when $R_0 > 1$, the disease will persist in the population and hence become endemic. The model has two non-negative equilibria, namely, the disease – free equilibrium and the endemic equilibrium. The existence and stability of the disease – free and endemic equilibria of the model and sensitivity analysis of the model were performed. The Herd immunity threshold, H_1, which shows the percentage or proportion of the population that needs to be immunized to control the transmission of the disease when there is an outbreak, was also considered.

From the results, the basic reproductive number for the SEIR model was estimated as $R_0 = 1.09305$. As indicated, $R_0 > 1$ and this means that, the disease will spread in case there is an outbreak. Incidentally, an average of one tuberculosis patient contacts 1.1706 susceptible individuals during an infectious period.

The steady states of the two non – negative equilibria were also found and analyzed. The disease – free equilibrium, $(s^*, e^*, i^*) = (1, 0, 0)$, was found to be unstable which is a saddle point, whiles the endemic equilibrium, $(s_e, e_e, i_e) = (0.914871, 0.00042479, 0.00139106)$, was also found to be asymptotically stable. These equilibrium points and analyses showed that the presence of an infected individual in a susceptible population will result in an outbreak of the disease.

In the sensitivity analysis, the relationship between the inputs and output variables of the model was undertaken. It was checked by looking at how changes in the transmission rate (β) and the recovery rate (γ) affects the endemicity of the disease – free equilibrium and the endemic equilibrium. In this case, the basic reproductive number, R_0 was used.

It was seen that, an increase in the transmission rate or a decrease in the recovery rate will make $R_0 > 1$ and the disease – free equilibrium is unstable. This indicates that, the disease will spread when there is an outbreak. Apparently, as the transmission rate decreased or the recovery rate increased, the result was $R_0 < 1$and hence the disease – free equilibrium is also asymptotically stable. This therefore means that the disease will not spread.

Also, when the transmission rate was increased or the recovery rate decreased, the result was $R_0 >$ 1and hence the endemic equilibrium is stable. This means that the disease will spread when there is an outbreak. On the other hand, as the transmission rate decreases or the recovery rate increases, $R_0 < 1$, and the endemic equilibrium is in an unstable state. This therefore means that the disease will not spread.

It was also found in the model that, increasing the number of infective individuals reduces the number of susceptibles whiles the number of exposed individuals also increases. These conclude that, if stakeholders fail to put in place proper measures to control and eradicate the disease, it will spread. Hence, immunization programmes as well as education on tuberculosis must be intensified throughout the communities and the country as a whole so that the disease can be curbed down.

From the paper, the herd immunity threshold was estimated to be 0.0851. This means that, about 8.51% of the population need to be immunized in order to control the spread of the disease. This is can be effective when early detections are reported at tuberculosis clinics for appropriate treatments. Children should also be vaccinated with Bacillus Calmette–Guérin (BCG) vaccines. This will help effectively disseminate the disease in them. When the percentage of immuned individuals exceed the herd immunity, the disease will fail to spread. Hence 8.51% shows the minimum percentage of the population that must be screened and immunized on regular basis in order for the disease not to spread in the district.

From the simulations, it was found out that, an increase in the number of immunized individuals in the population will also increase the level of immunity. This is to say that, if immunization are done on regular basis, especially among children, it will help increase the number of immuned individuals thereby decreasing the spread of tuberculosis within the community. From this, we can conclude that all children must be vaccinated to avoid making them exposed, because every child that is not immunized increases the number of exposed individuals in the system and this puts a threat in the district.

5. Conclusion

In search for the possible way of eradicating tuberculosis, there is the need to address the issue of the mechanism of the transmission and prevalence of the disease.

Many communicable diseases have been modeled using differential equations. The purpose of this work was to examine in detail, a mathematical models for the transmission and prevalence of tuberculosis and then to solve them using differential equations. The following assumptions were made before the model was formulated.

The model also pointed out that early detection has a positive impact on the reduction of tuberculosis transmission; that is there is a need to detect new cases as early as possible so as to provide early treatment for the disease. More people should be educated in order to create awareness to the disease transmission so that society will be aware of this deadly disease.

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Competing interests

Authors have declared that no competing interests exist.

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APPENDIX I

M-File for tuberculosis model

```
function dy=model(t,y,beta,gamma,mu,epsilon)
N= 124507;
dy=zeros(4,1);
dy(1)= mu*N-mu*y(1)-((beta*y(1)*y(3))/N);
dy(2)= ((beta*y(1)*y(2))/N)-(mu+epsilon)*y(2);
dy(3)= epsilon*y(2)-(mu+gamma)*y(3);
dy(4)= gamma*y(3)-mu*y(4);
```

Scripts Used in Calling the M-File for Tuberculosis Model

```
mu = 0.00875;
epsilon = 0.1666;
beta = 0.5853;
gamma = 0.5;
N= 124705;
options = odeset('RelTol',1e-9,'AbsTol',1e-9); [T,Y] = ode45(@emmaseir,[0 12],[390 100 10
0],options,beta,gamma,epsilon,mu);
figure(1)
plot(T,Y(:,1),'.')
legend('SUSCEPTIBLE')
xlabel('Time(Months)');ylabel('POPULATION OF SUSCEPTIBLE');
figure(2)
plot(T,Y(:,2),'.')
legend('EXPOSED')
xlabel('Time(Months)');ylabel('POPULATION OF EXPOSED');
figure(3)
plot(T,Y(:,3),'.')
legend('INFECTED')
xlabel('Time(Months)');ylabel('POPULATION OF INFECTED');
figure(4)
plot(T,Y(:,4),'.')
legend('REMOVED')
xlabel('TIME (MONTHS)');ylabel('POPULATION OF REMOVED');
```

APPENDIX II

Amansie West District Health Directorate
Tuberculosis (TB) Cases Detected and Recorded

Year	Number of cases
2002	48
2003	55
2004	58
2005	55
2006	23
2007	34
2008	62
2009	59
2010	62
2011	70

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www.sciencedomain.org/review-history.php?iid=313&id=6&aid=2449